NUMERICAL STUDY OF VOLTERRA DIFFERENCE EQUATIONS OF THE SECOND KIND

By

Said Ali Al-Garni

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

In Partial Fullfillment of the Requirements
for the degree
Master of Science
IN
Mathematical Sciences

KING FAHD UNIVERSITY
OF PETROLEUM & MINERALS
Dhahran, Saudi Arabia
April, 2003

KING FAHD UNVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **Said Ali Al-Garni** under the direction of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements of the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

Thesis Committee

Dr. Haydar Akca, Thesis Advisor

Dr. Basem Attili, Member

Dr. Abdulaziz Al-Shuaibi, Member

Dr. Khaled Furati
Department Chairman

Prof. Osama Jannadi
Dean of Graduate Studies

Date

Dedication

I wish to dedicate this thesis to my parents, my brothers, my sisters, my wife, my kids and my friends

Acknowledgment

First of all I thank "ALLAH" The Almighty for the knowledge, help and guidance HE has showered on me. Thanks are also to our Prophet Mohammad (pbuh) who encouraged us as Muslims to seek knowledge and stressed that science and Islam are never separable.

My sincere appreciations are due to Prof. Hydar Akca, my main advisor, for his help, support and encouragement in making this work possible. I am sincerely grateful to my thesis members, Dr. Basem Attili and Dr. Abdulaziz Al-Shuaibi for their helpful comments.

Acknowledgments are due to King Fahd University of Petroleum and Minerals for giving me the opportunity to pursue my study. A special thank is due to Prof. Ali Al-Daffa' and Prof Mohammad Al-Bar for their support and help.

My sincere appreciation are due to Dr. Salim Masoudi, my academic advisor, Dr. Khaled Furati, chairman, and all other faculty of the department of mathematical sciences for their concern and encouragement.

Last but not least, I put in record myhigh appreciation to the support of my parents, my brothers, my sisters, my wife, and my kids, Mohammad and Raghd, during the study period.

Contents

List of Tables

Thesis Abstract

Full Name Of Student	**:**	**Said Ali Al-Garni**
Title Of Study	**:**	**Numerical Study of Volterra Difference Equations of the Second Kind**
Major Field	**:**	**Mathematics**
Date Of Degree	**:**	**April 2003**

In this thesis, we will consider Volterra difference equations of the second kind. Three different kernels, nonconvolution, convolution and degenerate , are introduced. Boundedness, periodicity, and stability are among the most properties of Volterra difference equations discussed. We consider discretization of Volterra integral and integro-differential equations using numerical methods. Specific programs are constructed to solve these equations using numerical methods. Finally, we give estimations of solutions of both linear and nonlinear Volterra difference equations.

Preface

Volterra difference equations, whose solution is defined by the whole previous history, are widely used in the modeling of the processes in so many fields. In addition to networks, Volterra equation has been successfully applied to specific problems in such areas as communication, fluid mechanics, biophysics, optics, biomedical modeling, ecology (population dynamics), naval architecture, solid state, device modeling, problem of control, biomechanics, combinatorics, epidemics and some schemes of numerical solutions of integral and intgro-differential equations.

The work is organized as follows. Chapter 1 introduces the relevant basic definitions and notations that are commonly used for linear difference equations. In addition, some needed results are presented. Chapter 2 introduces Volterra difference equations. The scalar case, equations of nonconvolution, equations of convolution and equations of degenerate kernels are considered. Also, boundedness, periodicity, and stability are among the most properties of Volterra difference equations will be discussed. Moreover, Chapter 2 includes some results of the nonlinear case. Next, in Chapter 3, we consider discretization of Volterra integral and integro-differential equations using numerical methods such as backward Euler, trapezoidal and linear multistep methods. Specific programs are provided to solve these equations using numerical methods. Finally, in Chapter 4, we discuss estimations of solutions of both linear and nonlinear Volterra difference equations.

Chapter 1

Linear Difference Equations

Mathematical computions frequently are based on equations that allow us to compute the value of a function recursively from a given set of values. Such equations are called “difference equations”. These equations occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields. For example, if a certain population has discrete generations, the relation expresses itself in the difference equation

$$x(n+1) = f(x(n)), \tag{1.1}$$

where $x(n)$ is the size of population at the n the stage and one may consider n is a year. We may use this example to introduce some notations. Starting from a point x_0 , one may generate the sequence

$$\begin{aligned} & x_0,\ f(x_0),\ f(f(x_0)),\ f(f(f(x_0))),\ .... \qquad (1.2)\\ = \quad & x_0,\ f(x_0),\ f^2(x_0),\ f^3(x_0),\ ...\\ = \quad & x_0,\ x_1,\ x_2,\ x_3,\ .... \end{aligned}$$

This iterative procedure is an example of a discrete dynamical system. Letting

$x(n) = f^n(x_0)$, we have

$$x(n+1) = f^{n+1}(x_0) = f[f^n(x_0)] = f(x(n)).$$

Observe that $x(0) = f^0(x_0) = x_0$. If the function f in (1.1) is replaced by a function g of two variables, that is $g : Z^+ \times R \to R$, where Z^+ is the set of positive integers and R is the set of real numbers, then we have

$$x(n+1) = g(n, x(n)). \tag{1.3}$$

Equation (1.3) is called nonautonomous or time-variant whereas (1.1) is called autonomous or time-invariant. If an initial condition $x(n_0) = x_0$ is given, then for $n \geq n_0$ there is a unique solution $x(n) = x(n, n_0, x_0)$ of (1.3) such that $x(n_0, n_0, x_0) = x_0$. This may be shown easily by iteration

$$\begin{aligned}
x(n_0+1, n_0, x_0) &= g(n_0, x(n_0)) = g(n_0, x_0); \\
x(n_0+2, n_0, x_0) &= g(n_0+1, x(n_0+1)) = g(n_0+1, f(n_0, x_0)); \\
x(n_0+3, n_0, x_0) &= g(n_0+2, x(n_0+2)) = g[n_0+2, f(n_0+1, f(n_{0,}x_0))].
\end{aligned}$$

And inductively we get

$$x(n, n_0, x_0) = g[n-1, x(n-1, n_0, x_0)].$$

1.1 Linear First Order Difference Equations

A typical linear homogeneous first-order equation is given by

$$x(n+1) = a(n)x(n), \qquad x(n_0) = x_0 \qquad \text{for } \; n \geq n_0, \tag{1.4}$$

and the associated nonhomogeneous equation given by

$$y(n+1) = a(n)y(n) + g(n), \qquad y(n_0) = y_0 \qquad \text{for } n \geq n_0 \geq 0, \tag{1.5}$$

where in both equations, it is assumed that $a(n) \neq 0$, $a(n)$, *and* $g(n)$ are real-valued functions defined for $n \geq n_0 \geq 0$. One may find the solution of (1.4) by simple iteration.

Definition 1.1.1 *A point x^* in the domain of f is said to be an equilibrium point of (1.1) if it is a fixed point of f, i.e., $f(x^*) = x^*$.*

Definition 1.1.2

(i) The equilibrium point x^* of (1.1) is stable if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies $|f^n(x_0) - x^*| < \varepsilon$ for all $n > 0$. If x^* is not stable, then it is called unstable.

(ii) The point x^* is an asymptotically stable (attracting)equilibrium point if it is stable and there exists $\eta > 0$ such that $|x(0) - x^*| < \eta$ implies $\lim_{n\to\infty} x(n) = x^*$. If $\eta = \infty$, x^* is said to be globally asymptotically stable.

Definition 1.1.3 *Let b be in the domain of f Then;*

(i) b is called a periodic point of (1.1) if for some positive integer k, $f^k(b) = b$. Hence a point is k-periodic if it is a fixed point of f^k , that is, if it is an equilibrium point

of the difference equation

$$x(n+1) = g(x(n)), \tag{1.6}$$

where $g = f^k$;

(ii) b *is called eventually* k *periodic if for some positive integer* m, $f^m(b)$ *is a* k *periodic point. In other words,* b *is eventually* k*-periodic if* $f^{m+k}(b) = f^m(b)$ *.*

1.2 Linear Difference Equations of Higher Order

The normal form of a k th-order nonhomogeneous linear difference equation is given by

$$y(n+k) + p_1(n)y(n+k-1) + \cdots + p_k(n)y(n) = g(n)\ , \tag{1.7}$$

where $p_i(n)$ and $g(n)$ are real valued functions defined for $n \geq n_0$ and $p_k(n) \neq 0$ for all $n \geq n_0$. If $g(n)$ is identically zero then (1.7) is said to be a homogeneous equation. By letting $n = 0$ in (1.7), we obtain $y(k)$ in terms of $y(k-1), y(k-2), ..., y(0)$. Explicitly, we have

$$y(k) = -p_1(0)y(k-1) - p_2(0)y(k-2) - \cdots - p_k y(0) + g(0).$$

Once $y(k)$ is computed, we can find $y(k+1)$. Repeating this process, it is possible to evaluate all values of $y(n)$ for $n \geq k$.

A sequence $\{y(n)\}_{n_0}^{\infty}$ or simply $y(n)$ is said to be a solution of (1.7) if it satisfies the equation. Observe that if we specify the initial data of the equation, we are led to the corresponding initial value problem

$$y(n+k) + p_1(n)y(n+k-1) + \cdots + p_k(n)y(n) = g(n), \tag{1.8}$$

$$y(n_0) = a_0,\ y(n_0+1) = a_1,\ ....,\ y(n_0+k-1) = a_{k-1}, \tag{1.9}$$

where the $a_i \prime s$ are real numbers. In view of the above discussion, we conclude with the following result.

Theorem 1.2.1 *The initial value problems (1.8) and (1.9) have a unique solution* $y(n)$.

Proof. *see [1], [13], [24], and [31].* ■

Let us study the general theory of the k th-order linear homogeneous difference equations of the form

$$x(n+k) + p_1(n)x(n+k-1) + \cdots + p_k(n)x(n) = 0. \tag{1.10}$$

Definition 1.2.1 *The functions* $f_1(n),\ f_2(n),\ ...,\ f_r(n)$ *are said to be linearly dependent for* $n \geq n_0$ *if there are constants* $a_1,\ a_2,\ ....,\ a_r$ *not all zero, such that*

$$a_1 f_1(n) + a_2 f_2(n) + ...a_r f_r(n) = 0 \quad for \quad n \geq n_0.$$

Definition 1.2.2 *The functions* $f_1(n),\ f_2(n),\ ...,\ f_r(n)$ *are said to be linearly independent for all* $n \geq n_0$ *if* $a_1 f_1(n) + a_2 f_2(n) + ... + a_r f_r(n) = 0$ *whenever we have* $a_1 = a_2 = ... = a_r = 0 \quad for \quad n \geq n_0$.

Definition 1.2.3 *A set of* r *linearly independent solutions of (1.10) is called a fundamental set of solutions. The Casortian* $C(n)$ *of the solutions* $x_1(n),\ x_2(n),\ ...,$

$x_r(n)$ is given by

$$C(n) = \det \begin{pmatrix} x_1(n) & x_2(n) & \dots & x_r(n) \\ x_1(n+1) & x_2(n+1) & \dots & x_r(n+1) \\ \vdots & \vdots & \cdots & \vdots \\ x_1(n+r-1) & x_2(n+r-1) & \dots & x_r(n+r-1) \end{pmatrix}. \tag{1.11}$$

Theorem 1.2.2 *The set of solutions $x_1(n)$, $x_2(n)$, ..., $x_k(n)$ of (1.10) is a fundamental set if and only if for some $n_0 \in Z^+$, their Casoratian $C(n_0) \neq 0$.*

1.3 Linear Homogeneous Equations with Constant Coefficients

Consider the k th-order difference equation

$$x(n+k) + p_1 x(n+k-1) + \cdots + p_k x(n) = 0, \tag{1.12}$$

where the $p_i\prime s$ are constants, and $p_k \neq 0$. So, the procedure to find the general solution of (1.12) will be as follows. Suppose that the solutions of (1.12) are of the form λ^n, where λ_i is a complex number. After substituting into (1.12) we get

$$\lambda^k + p_1\ \lambda^{k-1} + \cdots + p_k = 0. \tag{1.13}$$

This is called the characteristic equation of (1.12) and $\lambda_i' s$ $(i = 1, 2, ..., k)$ are called the characteristic roots.

1.4 Linear Nonhomogeneous Equations

Consider (1.7) where $p_k(n) \neq 0$ for all $n \geq n_0$. The sequence $g(n)$ is called the forcing term, external force, the control, or input of the system.

Theorem 1.4.1 *Any solution $y(n)$ of (1.7) may be written as*

$$y(n) = y_p(n) + \sum_{i=1}^{k} a_i x_i(n),$$

where $\{x_1(n), x_2(n), ..., x_k(n)\}$ is a fundamental set of solutions of the homogeneous equation (1.10) and a solution of the nonhomogeneous equation (1.7) is called a particular solution and denoted by $y_p(n)$. General solution of the nonhomogeneous equation (1.7) as

$$y(n) = y_c(n) + y_p(n). \tag{1.14}$$

To compute the particular solution $y_p(n)$ the method of undetermined coefficients and variation of parameters are commonly using methods, see [1], [13], [24], and [31] for different forms of $g(n)$.

Example 1.4.1 *Consider the following equation*

$$y(n+2) + y(n+1) - 12y(n) = n2^n.$$

The characteristic roots of the homogneous equation are $\lambda_1 = 3$ and $\lambda_2 = -4$. Hence,

$$y_c(n) = c_1 3^n + c_2(-4)^n.$$

And the particular solution

$$y_p(n) = a_1 2^n + a_2 n 2^n.$$

Substituting this relation into our equation, we could find the values of a_1 *and* a_2*. Then the particular solution is*

$$y_p(n) = \frac{-5}{18} 2^n - \frac{1}{6} n 2^n,$$

and the general solution is

$$y(n) = c_1 3^n + c_2 (-4)^n - \frac{5}{18} 2^n - \frac{1}{6} n 2^n.$$

1.5 Systems of Difference Equations

Consider the following system of k linear first-order difference equations

$$\begin{aligned} x_1(n+1) &= a_{11}x_1(n) + a_{12}x_2(n) + ... + a_{1k}x_k(n) \\ x_2(n+1) &= a_{21}x_1(n) + a_{22}x_2(n) + ... + a_{2k}x_k(n) \\ &\vdots \\ x_k(n+1) &= a_{k1}x_1(n) + a_{k2}x_2(n) + ... + a_{kk}x_k(n) \end{aligned} \quad . \tag{1.15}$$

This system may be written in the vector form

$$\mathbf{x(n+1)} = A\mathbf{x(n)}, \tag{1.16}$$

where $\mathbf{x(n)} = (x_1(n), x_2(n), ..., x_k(n))^T \in R^k$, and $A = (a_{ij})$ is a $k \times k$ real nonsingular matrix. Since A does not depend on n so we called this system autonomous or time- invariant system. If for some $n_0 \geq 0$, $x(n_0) = x_0$ is specified, then system (1.16) is called an initial value problem. Moreover, by simple iteration, one may show that the solution is given by

$$x(n_0, n_0, x_0) = A^{n-n_0} x_0, \tag{1.17}$$

where $A^0 = I$, is the $k \times k$ identity matrix. Consider the system of the form

$$\mathbf{x(n+1)} = A(n)\mathbf{x(n)}, \tag{1.18}$$

where $A(n) = (a_{ij}(n))$ is $k \times k$ nonsingular matrix function. This is a homogeneous linear difference system which is nonautonomous or time-variant. The corresponding nonhomogeneous system is given by

$$\mathbf{x(n+1)} = A\mathbf{x(n)} + \mathbf{g(n)}, \tag{1.19}$$

where $\mathbf{g(n)} \in R^k$. We now establish the existence and uniqueness of solutions of (1.18).

Theorem 1.5.1 *For each $x_0 \in R^k$ and $n_0 \in Z^+$, there exists a unique solution $x(n, n_0, x_0)$ of (1.18) with $x(n_0, n_0, x_0) = x_0$.*

Let $\Phi(n)$ be a $k \times k$ matrix whose columns are solutions of (1.18) and denoted by

$$\Phi(n) = [x_1(n), x_2(n), ..., x_k(n)] \,.$$

Then $\mathbf{\Phi(n)}$ satisfies the difference equation

$$\mathbf{\Phi(n+1)} = A(n)\mathbf{\Phi(n)}. \tag{1.20}$$

Definition 1.5.1 *If $\Phi(n)$ is a matrix that is nonsingular for all $n \geq n_0$ and satisfies (1.20), then it is said to be a fundamental matrix for the system of equation (1.18).*

Theorem 1.5.2 *Matrix equation (1.20) has a unique solution $\Psi(n)$ with $\Psi(n_0) = I$.*

Corollary 1.5.1 *The unique solution of $x(n, n_0, x_0)$ of (1.18) with $x(n_0, n_0, x_0) = x_0$ is given by*

$$x(n, n_0, x_0) = \Phi(n, n_0)x_0. \tag{1.21}$$

If $x_1(n), x_2(n), ..., x_k(n)$ are solutions of (1.18), then $x(n) = \sum_{i=1}^{k} c_i x_i(n)$ is also a solution of (1.18)

Definition 1.5.2 *Assuming that $\{x_i(n)|1 \leq i \leq k\}$ is any linearly independent set of solutions of (1.18) defined to be*

$$x(n) = \sum_{i=1}^{k} c_i x_i(n), \tag{1.22}$$

where $c_i \in R$ and at least one $c_i \neq 0$.

Then formula (1.22) may be written as

$$x(n) = \Phi(n)c, \tag{1.23}$$

where $\Phi(n) = [x_1(n), x_2(n), ..., x_k(n)]$ is a fundamental matrix, and $c = (c_1, c_2, ..., c_k)^T \in R^k$. Now we will focus on the nonhomogeneous system (1.19) and define a particular solution $y_p(n)$ of (1.19).

Theorem 1.5.3 *Any solution $y(n)$ of (1.19) can be written as*

$$y(n) = \Phi(n)c + y_p(n), \tag{1.24}$$

for an appropriate choice of the vector c.

Lemma 1.5.1 *A particular solution of (1.19) can be given by*

$$y_p(n) = \sum_{r=n_0}^{n-1} \Phi(n, r+1)g(r), \tag{1.25}$$

with $y_p(n_0) = \mathbf{0}$.

Theorem 1.5.4 *(Variation of Constant Formula). The unique solution of the initial value problem*

$$y(n+1) = A(n)y(n) + g(n), \qquad y(n_0) = y_0, \tag{1.26}$$

is given by

$$y(n, n_0, y_0) = \Phi(n, n_0)y_0 + \sum_{r=n_0}^{n-1} \Phi(n, r+1)g(r). \tag{1.27}$$

or, more explicitly, by

$$y(n, n_0, y_0) = \left(\prod_{i=n_0}^{n-1} A(i)\right) y_0 + \sum_{r=i_0}^{n-1} \left(\prod_{i=r+1}^{n-1} A(i)\right) g(r).$$

Proof of the theorem follows immediately from Theorem 1.5.3 and Lemma 1.5.1.

Example 1.5.1 *Consider the following system*

$$y(n+1) = Ay(n) + g(n),$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \qquad g(n) = \begin{pmatrix} n \\ 1 \end{pmatrix}, \qquad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using the Putzer algorithm, see for detail [13], [24], and [31], one may show that

$$A^n = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}.$$

Hence

$$\begin{aligned} y(n) &= \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{r=0}^{n-1} \begin{pmatrix} 2^{n-r-1} & (n-r-1)\,2^{n-r-2} \\ 0 & 2^{n-r-1} \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2^n \\ 0 \end{pmatrix} + \sum_{r=0}^{n-1} \begin{pmatrix} r2^{n-r-1} + (n-r-1)\,2^{n-r-2} \\ 2^{n-r-1} \end{pmatrix} \\ &\vdots \\ &= \begin{pmatrix} 2^n + n2^{n-1} - \frac{3}{4}n \\ 2^n - 1 \end{pmatrix}. \end{aligned}$$

1.6 Stability Theory

First we introduce the commonly used notation of norms of vector $\mathbf{x} \in R^k$ and matrix $A \in R^{k\times k}$. These are

(i) the L_1 norm, $\|x\|_1 = \sum_{i=1}^{k} |x_i|$ and $\|A\|_1 = \max_{1\le j\le k} \sum_{i=1}^{k} |a_{ij}|$;

(ii) the L_∞ norm, $\|x\|_\infty = \max_{1\le i\le k} |x_i|$ and $\|A\|_\infty = \max_{1\le i\le k} \sum_{j=1}^{k} |a_{ij}|$;

(iii) the l_2 norm, $\|x\|_2 = \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}$ and $\|A\|_2 = \left[\rho(A^T A)\right]^{\frac{1}{2}}$,

where $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$, is the spectral radius of A. Consider the vector difference equation

$$x(n+1) = f(n, x(n)), \qquad x(n_0) = x_0, \tag{1.28}$$

where $\mathbf{x(n)} \in R^k$, $\mathbf{f} : Z^+ \times R^k \to R^k$. It is said to be *periodic* if for all $n \in Z$,

$$f(n+N, x) = f(n, x),$$

for some positive integer N. A point x^* in R^k is called an *equilibrium* point of (1.28) if

$$f(n, x^*) = x^*,$$

for all $n \geq n_0$. In most of the literature x^* is assumed to be the origin 0 and is called the zero solution.

Definition 1.6.1 *The equilibrium point x^* of (1.28) is said to be*

(i) stable if given $\varepsilon > 0$ and $n_0 \geq 0$ there exists $\delta = \delta(\varepsilon, n_0)$ such that $\|x_0 - x^*\| < \delta$ implies

$$\|x(n, n_0, x_0) - x^*\| < \varepsilon \quad \textit{for all} \quad n \geq n_0;$$

(ii) uniformly stable if δ may be chosen independent of n_0;

(iii) attractive if there exists $\mu = \mu(n_0)$ such that $\|x_0 - x^*\| < \mu$ implies

$$\lim_{n\to\infty} x(n, n_0, x_0) = x^*;$$

(iv) uniformly attractive if μ *may be chosen independent of* n_0;

(v) asymptotically stable if it is stable and attractive;

(vi) uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

In the following result we express the conditions for stability in terms of a fundamental matrix $\Phi(n)$ of system (1.18).

Theorem 1.6.1 *Consider system (1.18). Then its zero solution is*

(i) stable if and only if there exists a positive constant M *such that*

$$\|\Phi(n)\| \leq M \qquad \textit{for} \quad n \geq n_0 \geq 0;$$

(ii) uniformly stable if and only if there exists a positive constant M *such that*

$$\|\Phi(n,m)\| \leq M \quad \textit{for} \;\; n_0 \leq m \leq n < \infty;$$

(iii) asymptotically stable if and only if $\lim_{n\to\infty} \|\Phi(n)\| = 0$;

(iv) uniformly asymptotically stable if and only if there exists a positive constant M *and* $\eta \in (0,1)$ *such that*

$$\|\Phi(n,m)\| \leq M\eta^{n-m} \quad \textit{for } n_0 \leq m \leq n < \infty,$$

where $\Phi(n)$ *is the fundamental matrix.*

We know give a simple but powerful criterion for uniform stability and uniform asymptotic stability.

Theorem 1.6.2 *We have*

(i) If $\sum_{j=1}^{k} |a_{ij}(n)| \leq 1,\ 1 \leq j \leq k,\ n \geq n_0,$ *then the zero solution of (1.18) is uniformly stable.*

(ii) If $\sum_{j=1}^{k} |a_{ij}(n)| \leq 1 - \nu,$ *for some* $\nu > 0$, $1 \leq j \leq k,\ n \geq n_0,$ *then the zero solution of (1.18) is uniformly asymptotically stable.*

1.7 Liapunov′s Direct or Second Method

In his famous memoir, published in 1892, Russian mathematician A. M. Liapunov introduced a new method for investigating the stability of differential equations. This method, known as Liapunov' s Direct Method, allows one to investigate the qualitative nature of solutions without actually determining the solutions themselves see [9], [28]. Therefore, we regard it as one of the major tools in stability theory. The method hinges upon finding certain real-valued functions which are named after Liapunov. Adaptation of the Liapunov' s direct method to difference equations is as follows. We start with the autonomous difference equation (1.1) where $f : G \to R^k$, $G \subset R^k$, is continuous. Assume that x^* is an equilibrium point of (1.1), that is, $f(x^*) = x^*$.

Let $V : R^k \to R$ be defined as a real-valued function. The variation of V relative to (1.1) would be defined as

$$\Delta V(x) = V(f(x)) - V(x)$$

and

$$\Delta V(x(n)) = V(f(x(n))) - V(x(n)) = V(x(n+1)) - V(x(n)).$$

Notice that if

$$\Delta V(x) \leq 0,$$

V is nonincreasing along the solution of (1.1). The function V is said to be a *Liapunov function* on a subset $H \subset R^k$ if

(i) V is continuous on H and

(ii) $\Delta V(x) \leq 0$ whenever x and $f(x) \in H$.

Let $B(x, \gamma)$ denote the open ball in R^k of radius γ and center x defined by

$$B(x, \gamma) = \left\{ y \in R^k | \, \|y - x\| < \gamma \right\}.$$

We say that the real-valued function V is *positive definite* at x^* if

(i) $V(x^*) = 0$ and

(ii) $V(x) > 0$ for all $x \in B(x^*, \gamma)$, for some $\gamma > 0$.

Theorem 1.7.1 *(Liapunov Stability Theorem). If V is a Liapunov function for (1.1) on a neighborhood H of the equilibrium point x^*, and V is positive definite with respect to x^*, then x^* is stable. If, in addition, $\Delta V(x) < 0$, whenever $x, f(x) \in H$ and $x \neq x^*$, then x^* is asymptotically stable. Moreover, if $G = H = R^k$ and $V(x) \to \infty$, as $\|x\| \to \infty$ then x^* is globally asymptotically stable.*

To show how this theorem could be used, let us see the following example.

Example 1.7.1 *Consider this difference equation*

$$x(n+1) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} x(n).$$

Here we define V *on* R^2 *by*

$$V\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1^2 + x_2^2.$$

Then $V\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 0$, $V(x) > 0$ *otherwise, and*

$$\begin{aligned} \triangle V_n &= V\left(\begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{bmatrix}\right) - V\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\ &= (x_1\cos\theta + x_2\sin\theta)^2 + (-x_1\sin\theta + x_2\cos\theta)^2 - x_1^2 - x_2^2 \\ &= 0. \end{aligned}$$

Consequently, V *is a Liapunov function and the origin is stable.*

Investigation of the difference equation by Liapunov method is so called time domain analysis. Another way of saying difference equations investigated transforming into another domain. We may investigate difference equation without transforming into another domain. So the difference equations investigated as it is. This method is so called Z-transform.

1.8 Z - Transform Method

By using suitable transform, one may reduce the study of a linear difference or differential equation to examining an associated complex function. For example, Laplace transform method is widely used in solving and analyzing constant coefficients linear differential equations and continuous control systems, while the Z-transform is most suitable for linear difference equations and discrete systems. It is widely used in the analysis and design of digital control, communication, and signal processing see [22], [36]. The Z-transform technique is not new and may be traced back to De Moivre around the year 1730.

The Z-transform of a sequence $x(n)$, which is identically zero for negative integers n is defined

$$\widetilde{x}(z) = Z(x(n)) = \sum_{j=0}^{\infty} x(j) z^{-j}, \tag{1.29}$$

where z is a complex number. The set of numbers z in the complex plane for which (1.29) converges is called the region of convergence of $\widetilde{x}(z)$. The most commonly used method to find the region of convergence of (1.24) is the ratio test.

Let us discuss properties of the Z-transform.

(i) Linearity, let $\widetilde{x}(z)$ be the Z-transform of $x(n)$ with radius of convergence R_1 and let $\widetilde{y}(z)$ be the Z-transform of $y(n)$ with radius of convergence R_2. Then for any complex numbers α, β we have

$$Z\left[\alpha x(n) + \beta y(n)\right] = \alpha \widetilde{x}(z) + \beta \widetilde{y}(z),$$

for some $|z| > \max(R_1, R_2)$.

(ii) Shifting , let R be the radius of convergence of $\widetilde{x}(z)$. Then

(a) Right-shifting

$$Z\left[x(n-k)\right] = z^{-k} \widetilde{x}(z),$$

for $|z| > R,$ where $x(-i) = 0$ for $i = 1, 2, ..., k$.

(b) Left-shifting

$$Z\left[x(n+k)\right] = z^{k} \widetilde{x}(z) - \sum_{r=0}^{k-1} x(r) z^{k-r},$$

for $|z| > R$.

(iii) Convolution, a convolution $*$ of two sequences $x(n)$, $y(n)$ is defined by

$$x(n) * y(n) = \sum_{j=0}^{n} x(n-j)\ y(j) = \sum_{j=0}^{n} x(j)\ y(n-j).$$

Moreover, we can obtain the sequence $x(n)$ from $\widetilde{x}(z)$ using a process called the inverse Z-transform. This process is denoted by

$$Z^{-1}\left[\widetilde{x}(z)\right] = x(n).$$

Example 1.8.1 *Find the Z-transform of the sequence* $\{x(n) = a^n\}$.

$$\begin{aligned}\widetilde{x}(z) &= Z(x(n)) = \sum_{j=0}^{\infty} \frac{a^j}{z^j} = \sum_{j=0}^{\infty} \left(\frac{a}{z}\right)^k \\ &= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}, \qquad\qquad |z| > |a|.\end{aligned}$$

So far, we introduced basic theory and definitions of difference equations needed for the coming chapters.

Chapter 2

Volterra Difference Equations

Introduction

Volterra difference equation (VDE) of the second type is of the form

$$x(n+1) = Ax(n) + \sum_{j=0}^{n} B(n,j)x(j), \tag{2.1}$$

where A $\in$ R and B : $Z^+ \rightarrow$ R is a discrete function. This equation may be considered as the discrete analogue of the famous Volterra integro-differential equations

$$x'(t) = Ax(t) + \int_{0}^{t} B(t,s)x(s)ds. \tag{2.2}$$

Equation (2.1) has been widely used as a mathematical model in population dynamics. Both equations (2.1) and (2.2) represent a system in which the future state $x(n+1)$ does not depend only on the present state $x(n)$ but also on all past states $x(n-1), x(n-2), ..., x(0)$. Volterra difference equations (VDE) mainly arise in the modeling process of some real phenomena or by applying a numerical method to a Volterra integral or integro-differential equations. Actually much of their qualitative theory remains to be developed. Stability, boundedness, and periodicity are among the most important properties of the solution of Volterra difference equations see

[6]-[8], [21], [25]-[27], [29], [35], [43]. These properties will be covered in this chapter. For applications of the Volterra difference equations in combinatorics see [2], in Epidemics see [32], and in Numerical Methods for integro-differential equations see [8].

Moreover, there is another form which is the discrete analogue of the famous Volterra integral equations of the second type

$$x\ (t) = A + \int_0^t B(t,s)x(s)ds, \tag{2.3}$$

where A $\in$ R and B : $Z^+ \rightarrow$ R is a discrete function, which is

$$x(n) = A + \sum_{j=0}^{n} B(n,j)x(j). \tag{2.4}$$

We will discuss stability, asymptotic behavior, boundedness and existence of periodic solutions of Volterra difference equations.

2.1 The Scalar Case of Volterra Difference Equations of Convolution Type

Consider the convolution type of (2.1)

$$x(n+1) = Ax(n) + \sum_{j=0}^{n} B(n-j)x(j), \tag{2.5}$$

one of the most effective methods of dealing with (2.5) is the Z -transform method.

Let us rewrite (2.5) in the convolution form

$$x(n+1) = Ax(n) + B * x. \tag{2.6}$$

Taking the Z-transform of both sides of (2.6), we get

$$z\widetilde{x}(z) - zx(0) = A\widetilde{x}(z) + \widetilde{B}(z)\widetilde{x}(z),$$

which gives

$$\left[z - A - \widetilde{B}(z)\right]\widetilde{x}(z) = zx(0),$$

or

$$\widetilde{x}(z) = \frac{zx(0)}{\left[z - A - \widetilde{B}(z)\right]}. \tag{2.7}$$

Let

$$g(z) = z - A - \widetilde{B}(z). \tag{2.8}$$

The complex function $g(z)$ will play an important role in the stability analysis of Volterra difference equations. It plays the role of the characteristic polynomials of linear difference equations. In contrast to polynomials, the function $g(z)$ may have infinitely many zeros in the complex plane. The following lemma sheds some light on the location of the zeros of $g(z)$.

Lemma 2.1.1 *The zeros of*

$$g(z) = z - A - \widetilde{B}(z),$$

all lie in the region $|z| < c$, for some real positive constant c . Moreover, $g(z)$ has finitely many zeros z with $|z| \geq 1$.

Proof. Suppose that all the zeros of $g(z)$ do not lie in any region $|z| < c$ for any positive real number c . Then there exists a sequence $\{z_i\}$ of zeros of $g(z)$ with $|z_i| \to \infty$ as $i \to \infty$. Then

$$|z_i - A| = \left|\widetilde{B}\,(z_i)\right| \leq \sum_{n=0}^{\infty} |B(n)|\,|z_i|^{-n}\,. \tag{2.9}$$

Notice that the right-hand side of inequality (2.9) goes to $B(0)$ as $i \to \infty$, while the left-hand side goes to ∞ as $i \to \infty$, which is a contradiction. This proves the first part of the lemma.

To prove the second part of the lemma, we first observe from the first part that all zeros of $g(z)$ with $|z| \geq 1$ lie in the annulus $1 \leq |z| \leq c$ for some real number c . We may conclude that $g(z)$ is analytic in this annulus ($1 \leq |z| \leq c$). Therefore, $g(z)$ has only finitely many zeros in the region $|z| \geq 1$. ■

Theorem 2.1.1 *The zero solution of (2.5) is uniformly asymptotically stable if and only if*

$$z - A - \widetilde{B}(z) \neq 0, \qquad \text{for all } |z| \geq 1. \tag{2.10}$$

Proof. See [13]. ■

Since locating the zeros of $g(z)$ is impossible in most problems, to provide an explicit conditions for the stability of (2.5), we have the following theorem.

Theorem 2.1.2 *Suppose that $B(n)$ does not change sign for $n \in Z^+$. Then the zero solution of (2.5) is asymptotically stable if*

$$|A| + \left|\sum_{n=0}^{\infty} B(n)\right| < 1. \tag{2.11}$$

Proof. Let $\beta = \sum_{n=0}^{\infty} B(n)$ and $D(n) = \beta^{-1}B(n)$. Then $\sum_{n=0}^{\infty} D(n) = 1$. Furthermore , $\widetilde{D}(1) = 1$ and $\left|\widetilde{D}(z)\right| \leq 1$ for all $|z| \geq 1$. Let us rewrite $g(z)$ in the form

$$g(z) = z - A - \beta\widetilde{D}(z). \tag{2.12}$$

To prove the asymptotic stability of zero solution of (2.5), it suffices to show that $g(z)$ has no zero z with $|z| \geq 1$. So assume that there exists a zero z_r of $g(z)$ with $|z_r| \geq 1$. Then by (2.12) we obtain

$$|z_r - A| = \left|\beta\widetilde{D}(z)\right| \leq |\beta|\,. \tag{2.13}$$

Using condition (2.11) one concludes that

$$|z_r| \leq |A| + |\beta| < 1,$$

which is a contradiction. This concludes the proof of the theorem. ■

Theorem 2.1.3 *Suppose that B(n) does not change sign for n∈ Z^+ . Then the zero solution of (2.5) is not asymptotically stable if any one of the following conditions hold:*

$$\begin{aligned}
i)\ A + \sum_{n=0}^{\infty} B(n) &\geq 1. \\
ii)\ A + \sum_{n=0}^{\infty} B(n) &\leq -1 \textit{ and } B(n) > 0 \ \textit{ for some } n \in Z^+. \\
iii)\ A + \sum_{n=0}^{\infty} B(n) &< -1 \textit{ and } B(n) < 0 \ \textit{ for some } n \in Z^+,
\end{aligned}$$

$$\textit{and} \ \sum_{n=0}^{\infty} B(n) \textit{ is sufficiently small.}$$

Proof. See [1], [13], [24], and [31]. ■

The results of Theorem 2.1.2 are not extendable to uniform stability. To treat the problem of uniform stability, the use of Liapunov functionals produce interesting results.

Theorem 2.1.4 *The zero solution of (2.5) is uniformly stable if*

$$|A| + \sum_{j=0}^{n} |B(j)| \leq 1, \qquad \text{for all } n \in Z^+. \tag{2.14}$$

Proof. Let E be the space of all infinite sequences of complex numbers. For $x \in E$, let

$$V(x) = |x(n)| + \sum_{r=0}^{n-1} \sum_{s=n}^{\infty} |B(s-r)| \, |x(r)| \, . \tag{2.15}$$

Then

$$\begin{aligned} \triangle V(x) &= \left| Ax(n) + \sum_{j=0}^{n} B(n-j)x(j) \right| + \sum_{r=0}^{n} \sum_{s=n+1}^{\infty} |B(s-r)| \, |x(r)| - |x(n)| \\ &\quad - \sum_{r=0}^{n-1} \sum_{s=n}^{\infty} |B(s-r)| \, |x(r)| \qquad (2.16) \\ &\leq \left(|A| + \sum_{j=0}^{\infty} |B(j)| - 1 \right) |x(n)| \, . \qquad (2.17) \end{aligned}$$

By assumption (2.14), we thus have

$$\triangle V(x) \leq 0. \tag{2.18}$$

From (2.15) we obtain $|x(n)| \leq V(x)$. Using inequality (2.18) and expression (2.15) again we obtain

$$|x(n)| \leq V(x) \leq |x(0)| \, .$$

Consequently, the zero solution is uniformly stable. ■

This finished the introduction of the scalar case. Now we will deal with systems in the coming sections.

2.2 Equations of Nonconvolution Type

We consider the following system of difference equations of nonconvolution type

$$x(n+1) = A(n)\, x(n) + \sum_{r=0}^{n} B(n,r)\, x(r), \tag{2.19}$$

and its perturbation

$$y(n+1) = A(n)\, y(n) + \sum_{r=0}^{n} B(n,r)\, y(r) \, + g(n), \tag{2.20}$$

where $A(n), B(n,r)$ are $k \times k$ matrix functions on Z^+ and $Z^+ \times Z^+$, respectively, and $g(n)$ is a vector function on Z^+. We define the resolvent matrix $R(n,m)$ of (2.20) as the unique solution of the matrix difference equation

$$R(n+1,m) = A(n)R(n,m) + \sum_{r=m}^{n} B(n,r)\, R(r,m)\,, \qquad n \geq m, \tag{2.21}$$

with $R(m,m) = I$.

It is straight forward to show the existence and uniqueness of $R(n,m)$. Using the resolvent matrix $R(n,m)$, we can establish the following variation of constants formula.

Proposition 2.2.1 *(Variation of Constants Formula). The unique solution* $y(n,0,y_0)$ *of (2.20) satisfying* $y(0)=y_0$ *is given by*

$$y(n,0,y_0) = R(n,0)y_0 + \sum_{r=0}^{n-1} R(n,r+1)\, g(r). \tag{2.22}$$

Proof. Using (2.21), one may easily verify that formula (2.22) gives a solution of (2.20). ∎

Periodicity is one of the important property in the real life problems. Such as mathematical biology, medicine, population dynamics and other engineering problems, periodic solutions play vital role. Therefore, to find a periodic solution for the Volterra difference system with infinite delay, consider the equation of the form

$$z(n+1) = A(n)\, z(n) + \sum_{r=-\infty}^{n} B(n,r)\, z(r) \; + g(n), \tag{2.23}$$

where

$$A(n+N) = A(n), \qquad B(n+N, m+N) = B(n,m), \qquad g(n+N) = g(n), \tag{2.24}$$

for all n, $m \in Z$ and for some positive integer N . We assume that

$$\sum_{r=0}^{\infty} |B(n,n-r)| < \infty. \tag{2.25}$$

To define a solution of (2.23) we need to introduce the concept of an initial function. We define an initial function $\phi(n)$ of (2.23) as a function from Z^- to R^k. A solution $z(n,0,\phi)$ of (2.23) is a sequence that satisfies (2.23) for $n \in Z^+$ and $z(r) = \phi(r)$ for $r \in Z^-$. All initial functions in this work are assumed to be bounded. The existence and uniqueness of solutions of initial value problems of (2.23) may now be easily established using (2.25).

Obviously, (2.23) reduces to (2.20) if

$$\sum_{r=-\infty}^{-1} B(n,r)\,\phi(r) = 0,$$

or if we let $\phi(r) = 0$ for all negative integers r . The equation in the form (2.23), however, has the advantage over the form (2.20) in that if $z(n)$ is a solution of (2.23), then so is $z(n+N)$. And this is indeed a crucial requirement in our search for periodic solutions.

We are in a position to give a crucial lemma.

Lemma 2.2.1 *If $y(n)$ is a solution of (2.20) bounded on Z^+ , then there is a corresponding solution z(n) of (2.23) such that for every n = 0, ± 1, $\pm 2, \ldots$, $z(n)$ is the limit of some subsequence of $y(n)$.*

Proof. Let $y(n)$ be a bounded solution of (2.20). Then $\{y(rN)\}_{r=1}^{\infty}$, with N as defined in (2.24), is bounded and thus has a convergent subsequence $\{y(r_{i0}N)\}$ which converges to a point in R^k , say $z(0)$. There is a subsequence $\{r_{i1}N\}$ of $\{r_{i0}N\}$ such that both subsequences $\{y(1+r_{i1}N)\}$ and $\{y(-1+r_{i1}N)\}$ converge to $z(1)$ and $z(-1)$, respectively. Inductively, one may show that for each nonnegative integer n, $\left\{y\left[\pm(n-1)+r_{i(n-1)}N\right]\right\}$ converges to $z(n-1)$ and $z\left[-(n-1)\right]$, respectively, and $\{y(\pm n + r_{in}N)\}$ converges to $z(n)$ and $z(-n)$, respectively, where $\{r_{in}\}$ is a subsequence of $\left\{r_{i(n-1)}\right\}$.

We are now going to show that $\{z(n)\}_{-\infty}^{\infty}$ is actually a solution of (2.23). From

(2.20) we have

$$\begin{aligned} y\left[n+1+r_{i(n+1)}N\right] &= A(n)\,y\left[n+r_{i(n+1)}N\right] \\ &+ \sum_{j=0}^{n+r_{i(n+1)}N} B(n+r_{i(n+1)}N, j)\,y(j) + g(n+r_{i(n+1)}N) \end{aligned}$$

$$\begin{aligned} y\left[n+1+r_{i(n+1)}N\right] &= A(n)\,y\left[n+r_{i(n+1)}N\right] \\ &+ \sum_{j=-r_{i(n+1)}N}^{n} B(n,j)\,y(j+r_{i(n+1)}N) + g(n). \end{aligned} \tag{2.26}$$

As $r_{i(n+1)} \to \infty$, the left-hand side of (2.26) converges to $z(n+1)$ and the right-hand side of (2.26) converges to

$$A(n)\,z(n) + \sum_{j=-\infty}^{n} B(n,j)\,z(j) + g(n).$$

Therefore $z(n)$ is a solution of (2.23). ■

We need to state one more observation before the main result .

Lemma 2.2.2 *If* $A(n+N) = A(n)$, $g(n+N) = g(n)$, $B(n+N, m+N) = B(n,m)$ *for all* $n, m \in Z$ *and for some positive integer* N*, then*

$$R(n+N, m+N) = R(n,m) \tag{2.27}$$

Proof. The proof of (2.27) follows easily from (2.21). ■

Theorem 2.2.1 *Suppose that the zero solution of (2.19) is uniformly asymptotically stable. Then (2.23) has the unique N-periodic solution*

$$z(n) = \sum_{m=-\infty}^{n-1} R(n, m+1)g(m). \tag{2.28}$$

Proof. Since the zero solution of (2.19) is uniformly asymptotically stable, it follows that $|R(n,m)| \leq L\nu^{n-m}$ for some $L > 0$, $\nu \in (0,1)$ and thus $\lim_{n\to\infty} R(n,0) = 0$. Furthermore, using the variation of constants formula (2.22), one may show easily that solutions of (2.20) are bounded. Hence one may use Lemma 2.2.1 to construct a sequence $\{y(n + r_{in}N)\}$ from a bounded solution $y(n)$ of (2.20) such that $\{y(n + r_{in}N)\}$ converges to a solution $z(n)$ of (2.23). Formula (2.22) now gives

$$y(n + r_{in}N) = R(n + r_{in}N, 0)y_0 + \sum_{m=-r_{in}N}^{n-1} R(n, m+1)g(m).$$

Hence

$$z(n) = \lim_{r_{in}\to\infty} y(n + r_{in}N) = \sum_{m=-\infty}^{n-1} R(n, m+1)g(m).$$

By Lemma 2.2.2, $z(n)$ is N-periodic.

It remains to show that $z(n)$ in (2.28) is the only N-periodic solution of (2.23). To prove this, let us assume that there is another N-periodic solution $\hat{z}(n)$ of (2.23). Then $\psi(n) = z(n) - \hat{z}(n)$ is an N-periodic solution of the equation

$$\psi(n+1) = A(n)\psi(n) + \sum_{r=0}^{n} B(n,r)\psi(r) + \sum_{r=-\infty}^{-1} B(n,r)\psi(r).$$

Hence, by variation of constants formula (2.22) we have

$$\begin{aligned}\psi(n) &= R(n,0)\psi(0) + \sum_{j=0}^{n-1} R(n,j+1)\left[\sum_{r=-\infty}^{-1} B(j,r)\psi(r)\right] \\ &= R(n,0)\psi(0) + \sum_{j=0}^{n-1} R(n,j+1)\left[\sum_{r=j+1}^{\infty} B(j,j-r)\psi(j-r)\right]\end{aligned}$$

$$|\psi(n)| \le |R(n,0)|\,|\psi(0)| + M\sum_{j=0}^{n-1} |R(n,j+1)|\,.\sum_{r=j+1}^{\infty} |B(j,j-r)|\,,$$

where $M = sup\,\{|\psi(n)| \mid n \in Z\}$.

Hence $\lim_{n\to\infty}\psi(n) = 0$. Since $\psi(n)$ is periodic, it follows that $\psi(n)$ is identically zero. Consequently, $z(n) = \hat{z}(n)$. ■

Remark 2.2.1 *Suppose that system (2.20) has an N-periodic solution y(n) and z(n) is the unique N-periodic solution (2.28) of (2.23). Then* $\omega(n) = z(n) - y(n)$ *is a solution of the equation*

$$\omega(n+1) = A\,\omega(n) + \sum_{r=0}^{n} B(n,r)\omega(r) + \sum_{r=-\infty}^{-1} B(n,r)\phi(r), \tag{2.29}$$

where ϕ *is the initial function of* $z(n)$.

By the variation of constants formula (2.22) we have

$$\omega(n) = R(n,0)\omega(0) + \sum_{j=0}^{n-1} R(n,j+1)\left[\sum_{r=-\infty}^{-1} B(j,r)\phi(r)\right].$$

Since $\lim_{n\to\infty}\omega(n) = \lim_{n\to\infty}(z(n) - y(n)) = 0$. Consequently, $z(n) \equiv y(n)$. It follows from (2.29) that

$$\sum_{r=-\infty}^{-1} B(n,r)\phi(r) = 0.$$

2.3 Equations of Convolution Type

Consider the Volterra difference system of convolution type

$$x(n+1) = A\,x(n) + \sum_{r=0}^{n} B(n-r)\,x(r), \tag{2.30}$$

and its perturbation

$$y(n+1) = A\,y(n) + \sum_{r=0}^{n} B(n-r)\,y(r) + g(n), \tag{2.31}$$

where A is a $k \times k$ nonsingular matrix, $B(n)$ is a $k \times k$ matrix function, and $g(n)$ is a vector function in R^k . The existence and uniqueness of solutions (2.31) may be established by a straightforward argument. Let $e_i = (0, ..., 0, 1, 0, ..., 0)^T$ be the standard i th unit vector in R^k , $1 \leq i \leq k$. The $k \times k$ matrix $X(n)$, whose i th column is $x_i(n)$, is called the fundamental matrix of system (2.30). Notice that $X(0) = I$; the identity $k \times k$ matrix. Furthermore

$$x(n, 0, x_0) = X(n)x_0$$

is the unique solution of (2.30) with

$$x(0, 0, x_0) = x_0.$$

Moreover,

$$X(n+1) = A\,X(n) + \sum_{r=0}^{n} B(n-r)\,X(r). \tag{2.32}$$

The fundamental matrix $X(n)$ enjoys all the properties possessed by its counterpart in linear ordinary difference equations such as

$$\mathbf{x(n+1)} = A\mathbf{x(n)}.$$

We are now going to develop the Variation of Constants Formula for system (2.31). Recall that the Z-transform $Z[x(n)]$ of a sequence $x(n)$ is defined as

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n)\, z^{-n}, \qquad |z| > d,$$

where z is a complex number, and d is the radius of convergence of $Z[x(n)]$. A common notation for the Z-transform of $x(n)$ is $\widetilde{x}(z)$. The Z-transform $\widetilde{B}(z)$ of the matrix $B(n)$ is defined similarly. Obviously if

$$\sum_{n=0}^{\infty} |B(n)| < \infty,$$

then $\widetilde{B}(z)$ exists.

Theorem 2.3.1 *(Variation of Constants Formula for Convolution Type). Suppose that the Z-transform $\widetilde{B}(z)$ of $B(n)$ exists. Then the solution of (2.31) is given by*

$$y(n, 0, y_0) = X(n)y_0 + \sum_{r=0}^{n-1} X(n-r-1)g(r). \tag{2.33}$$

Proof. We first observe that

$$X(n+1) = A\,X(n) + \sum_{r=0}^{n} B(n-r)\,X(r). \tag{2.34}$$

Taking the Z-transform of the matrix equation (2.34), we obtain

$$z\widetilde{X}(z) - X(0)z = A\widetilde{X}(z) + \widetilde{B}(z)\widetilde{X}(z), \quad |z| > d.$$

Hence

$$\left[zI - A - \widetilde{B}(z)\right] \widetilde{X}(z) = zI, \qquad |z| > d. \tag{2.35}$$

Since the right-hand side of (2.35) is nonsingular, it follows that the left hand side $\left[zI - A - \widetilde{B}(z)\right]$ of (2.35) is also nonsingular. Thus

$$\widetilde{X}(z) = z\left[zI - A - \widetilde{B}(z)\right]^{-1}, \qquad |z| > d. \tag{2.36}$$

Taking the Z-transform of both sides of (2.31) we obtain

$$\widetilde{y}(z) = \left[zI - A - \widetilde{B}(z)\right]^{-1}(zy_0 + \widetilde{g}(z)), \qquad |z| > d.$$

Taking the inverse Z-transform of both sides of the above equation produces formula (2.33). ■

We remark here that the resolvent matrix $R(n,m)$ discussed in last part for equations of nonconvolution type is closely related to the fundamental matrix $X(n)$. By uniqueness of solutions, it is easy to see that $R(n,0) = X(n)$, and $R(n,m) = X(n-m)$, for equations of convolution type such as (2.30). Therefore, all the results of Section 2.2 extend here by simply replacing $R(n,m)$ by $X(n-m)$.

Example 2.3.1 *Consider the system*

$$y(n+1) = A\,y(n) + \sum_{r=0}^{n} B(n-r)\,y(r) + g(n), \quad x(0) = 0,$$

$$where\ A = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & -\sqrt{6} \end{pmatrix},\ B(n) = \begin{pmatrix} 2^{-n/2} & 0 \\ 0 & 6^{-n/2} \end{pmatrix}\ \ and\ \ g(n) = \begin{pmatrix} n \\ 0 \end{pmatrix}.$$

Then the fundamental matrix

$$X(n) = \begin{pmatrix} (1+\sqrt{2})\,2^{n-1} + \frac{(1-\sqrt{2})}{2}(-1)^n & 0 \\ 0 & \frac{(3-\sqrt{6})}{5}(3)^n + \frac{(2+\sqrt{6})}{5}(-2) \end{pmatrix},$$

and using Variation of Constants Formula (2.33) we can obtain

$$y(n,0,y(0)) = \begin{pmatrix} \frac{(1+\sqrt{2})}{2}(2^n - n - 1) + \frac{(1-\sqrt{2})}{8}\left[(-1^n) + 2n - 1\right] \\ 0 \end{pmatrix}.$$

2.4 A Criterion for Uniform Asymptotic Stability

We say that $x(n,n_0,\phi)$ is a solution of (2.3) with a bounded initial function $\phi : [0,n_0] \to R^k$ if it satisfies (2.19) for $n \geq n_0$ and agrees with ϕ on $[0,n_0]$. Then the zero solution of (2.19) is uniformly stable if for every $\varepsilon > 0$ and any $n_0 \in Z^+$, there exists $\delta > 0$ independent of n_0 such that $|x(n,n_0,\phi)| < \varepsilon,$ for all $n \geq n_0$, whenever $\|\phi\| < \delta,$ where $\|\phi\| = \sup\{|\phi| \mid 0 \leq n \leq n_0\}$. The zero solution of (2.19) is uniformly asymptotically stable if it is uniform stable and there is $\eta > 0$ such that for every $\varepsilon > 0$ there exists $N(\varepsilon)$ independent of n_0 , such that $|x(n,n_0,\phi)| < \varepsilon$ for all $n \geq n_0 + N(\varepsilon)$ whenever $\|\phi\| < \eta$.

It may be shown that the zero solution of (2.19) is uniformly asymptotically stable if and only if $|R(n,m)| \leq L\nu^{n-m},$ for some $L > 0$, $0 < \nu < 1$, and $n \geq m \geq 0$. This clearly implies that $\lim_{n\to\infty} R(n,0) = 0$.

Several criteria for the uniform asymptotic stability of the zero solution of (2.30) have been established. Recall that in (2.30) $A = (a_{ij})$ and $B(n) = (b_{ij}(n))$. Let

$$\beta_{ji} = \sum_{n=0}^{\infty} |b_{ji}(n)|\,.$$

Then the zero solution of (2.30) is uniformly asymptotically stable if for all i , $1 \leq i \leq k$, the following condition holds,

$$\sum_{j=1}^{k} \left[|a_{ji}| + \beta_{ji}\right] \leq 1 - \delta, \tag{2.37}$$

for some $\delta > 0$.

We are now going to extend this result to equations of nonconvolutions type (2.19).

$$|x| = \sum_{i=1}^{k} |x_i|\ ,\ \beta_{ji} = \sum_{s=n}^{\infty} |b_{ji}(s,n)| < \infty.$$

Let us assume here that for any $n_0 \geq n$,

$$\sup_{n_0 \geq 0} \sum_{r=0}^{n_0-1} \sum_{s=n_0}^{\infty} |b_{ij}(s,r)| < \infty, \qquad 1 \leq i,\ j \leq k. \tag{2.38}$$

Theorem 2.4.1 *Suppose that for* $1 \leq i \leq k$, $n \geq n_0$,

$$\sum_{j=1}^{k} \left[|a_{ji}(n)| + \beta_{ji}\right] \leq 1 - c \tag{2.39}$$

for some $c \in (0,1)$ *Then the zero solution of (2.19) is uniformly asymptotically stable.*

Proof. Define the Liapunov function

$$V(n,x(\cdot)) = \sum_{i=1}^{k}\left[|x_i(n)| + \sum_{j=1}^{k}\ \sum_{r=0}^{n-1}\sum_{s=n}^{\infty}|b_{ij}(s,r)|\,|x_j(r)|\right].$$

For equation (2.19) we have

$$\begin{aligned}\triangle V(n,x(\cdot)) \ &\leq \ \sum_{i=1}^{k}\left[\sum_{j=1}^{k}|a_{ij}(n)|\,|x_j(n)| - |x_i(n)| + \sum_{j=1}^{k}\sum_{s=n}^{\infty}|b_{ij}(s,n)|\,|x_j(n)|\right]\\ &\leq \ \sum_{i=1}^{k}\left[\sum_{j=1}^{k}|a_{ji}(n)|\,|x_i(n)| - |x_i(n)| + \sum_{j=1}^{k}\sum_{s=n}^{\infty}|b_{ji}(s,n)|\,|x_i(n)|\right]\end{aligned}$$

$$\leq \sum_{i=1}^{k}\left[\sum_{j=1}^{k}|a_{ji}(n)| + \beta_{ji}(n) - 1\right]|x_i(n)|. \tag{2.40}$$

Using (2.39) we conclude that

$$\triangle V(n,x(\cdot)) \ \leq 0. \tag{2.41}$$

Now using condition (2.38), one can obtain

$$V(n_0,\phi(\cdot)) = \sum_{i=1}^{k}\left[|\phi_i(n_0)| + \sum_{j=1}^{k}\sum_{r=0}^{n-1}\ \sum_{s=n}^{\infty}|b_{ij}(s,r)|\,\left|\phi_j(r)\right|\right] \leq \ \gamma\ \|\phi\|\,, \tag{2.42}$$

where $\|\phi\| = \sup\{|\phi(n)| \mid 0 \leq n \leq n_0\}$, and

$$\gamma = 1 + \sup_{n_0\geq 0}\sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{r=0}^{n-1}\ \sum_{s=n}^{\infty}|b_{ij}(s,r)|\,.$$

By (2.41) and (2.42) we have

$$|x(n, n_0, \phi)| \leq V(n, x(\cdot)) \leq V(n_0, \phi(\cdot)) \leq \gamma \, \|\phi\| \, .$$

Using condition (2.39) in (2.40) yields

$$\triangle V(n, x(\cdot)) \leq -c \, |x(n)| \, . \tag{2.43}$$

This implies that the zero solution of (2.19) is uniformly asymptotically stable. ■

Example 2.4.1 *Consider the system*

$$x(n+1) = A\, x(n) + \sum_{j=0}^{n} B(n-j)\, x(j) \, ,$$

$$\textit{where} \quad A = \begin{pmatrix} 0 & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}, \quad B(n) = \begin{pmatrix} 4^{-n-1} & 0 \\ 0 & 3^{-n-1} \end{pmatrix}.$$

Then

$$\beta_{11} = \sum_{n=0}^{\infty} \left| 4^{-n-1} \right| = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{3},$$

$$\beta_{12} = 0, \; \beta_{21} = 0,$$

$$\beta_{22} = \sum_{n=0}^{\infty} \left| 3^{-n-1} \right| = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{2}.$$

Now checking

$$\left[\sum_{j=1}^{k} \left[|a_{ji}(n)| + \beta_{ji} \right] \leq 1 - c \qquad \right]$$

1. *For i=1,* $\sum_{j=1}^{2}\left[|a_{j1}|+\beta_{j1}\right]\leq 1-c$,

$$|a_{11}|+|a_{21}|+\beta_{11}+\beta_{21}=0+\frac{1}{3}+\frac{1}{3}+0=\frac{2}{3}\leq 1-c.$$

2. *For i=2,* $\sum_{j=1}^{2}\left[|a_{j2}|+\beta_{j2}\right]\leq 1-c,$

$$|a_{12}|+|a_{22}|+\beta_{12}+\beta_{22}=\frac{1}{5}+\frac{1}{4}+0+\frac{1}{2}=\frac{19}{20}\leq 1-c.$$

So it is uniformly asymptotically stable.

2.5 Boundedness of Solutions of Volterra Equations

We will show that, in the particular case of linear homogeneous equations, boundedness is equivalent to stability. Let us consider the following linear homogeneous Volterra difference equation

$$y(i+1)=\sum_{j=i_0}^{i}A(i,j)y(j), \qquad i\geq i_0\ , y(i)\in R^n, A(i,j)\in R^{n\times n}. \tag{2.44}$$

Theorem 2.5.1 *A necessary and sufficient condition for boundedness (uniform boundedness) of system (2.45) is its stability (uniform stability) respectively.*

Proof. Assume that the system given by (2.45) be stable. It means, according to the stability definition, given $\varepsilon>0$ and $n_0\geq 0$ there exists $\delta=\delta(\varepsilon,i_0)>0$ such that $\|y_0\|<\delta(\varepsilon,i_0)$ implies $\|y(i,i_0,y_0)\|<\varepsilon$ for all $i\geq i_0$. Note that, due to the homogeneity of (2.45) for any constant $C > 0$ we get,

$$y(i, i_0, Cy_0) = Cy(i, i_0, y_0)\ . \tag{2.45}$$

Choose and fix now some $\varepsilon > 0$ and corresponding $\delta(\varepsilon, i_0)$ and let us take an arbitrary r . We want to prove that if $\|y_0\| \le r$ then there exists $\alpha(i_0, r)$ such that $\|y(i, i_0)\| < \alpha(i_0, r)$. Take $\alpha(i_0, r) = r\varepsilon/\delta(\varepsilon, i_0)$, then

$$\|y_0\| \le r \Longrightarrow \frac{\varepsilon}{\alpha}\|y_0\| < \delta(\alpha, i_0) \Longrightarrow \left\| y(i, i_0, \frac{\varepsilon y_0}{\alpha}) \right\| < \varepsilon \tag{2.46}$$

for the stability hypothesis. Then, from (2.46) there results $\|\ y(i, i_0, y_0)\| <\ \alpha(i_0, r)$. Hence the sufficiency of boundedness of (2.45) is proven.

Denote $R(i, j)$ as a fundamental matrix corresponding to system (2.45) and defined by the equation

$$R(i+1, j) = \sum_{l=j}^{i} A(i, l)R(l, j), \qquad R(j, j) = I\ ,\ i \ge j.$$

Because of the boundedness of (2.45) there exists a constant $\alpha(i_0) > 0$ such that

$$\|R(i, i_0)\| \le \alpha(i_0).$$

From this fact and variation of constant formula for (2.45) it follows that, for any

$$y_0, x_0 \in R^n, \|y(i, i_0, y_0) - x(i, i_0, x_0)\| = \|R(i, i_0)(y_0 - x_0)\| \le \alpha(i_0)\,\|y_0 - x_0\|\,. \tag{2.47}$$

This means the necessary condition of the stability of (2.45) is proven. ■

Now we introduce the Discrete Gronwall Inequality that helps to discus the boundedness of systems with perturbations.

Lemma 2.5.1 *(Discrete Gronwall Inequality). Let $n \in Z(n_0)^+$, $k(n) \geq 0$ and*

$$y(n+1) \leq y(n_0) + \sum_{s=n_0}^{n} \left[k(s)y(s) + p(s)\right].$$

Then,

$$y_n \leq y(n_0)\exp(\sum_{s=n_0}^{n-1} k(s)) + \sum_{s=n_0}^{n-1} p(s)\exp(\sum_{\tau=s+1}^{n-1} k(\tau)), \quad n \geq n_0. \tag{2.48}$$

Consider the system with disturbances

$$y(i+1) = \sum_{j=i_0}^{i} (A(i,j) + B(i,j))y(j), \tag{2.49}$$

and whether there are some conditions on the perturbation matrices $B(i,j)$ under which, from boundedness of (2.45), the boundedness of (2.50) can be proven. Denote by $\|B\|$ the norm of the matrix B , defined by the formula $\|B\| = \sup_{\|x\|=1} \|Bx\|$. Then the following theorems can be proven.

Theorem 2.5.2 *Assume system (2.45) is uniformly bounded and the matrices $B(i,j)$ satisfy the condition*

$$\sum_{j=i_0}^{\infty} \sum_{r=j}^{\infty} \|B(r,j)\| < \infty . \tag{2.50}$$

Then (2.50) is bounded.

Proof. The solution of (2.50) can be represented in the form

$$y(i) = R(i,i_0)y_0 + \sum_{r=i_0}^{i-1} R(i,r+1) \sum_{j=i_0}^{r} B(r,j)y(j), \qquad i > i_0. \tag{2.51}$$

Because of the uniform boundedness of (2.45) there exists a constant $C_1 > 0$ such that $\|R(i,i_0)\| \le C_1$. Consequently, taking into account relation (2.52), we have

$$\|y(i)\| \le C_1 \left[\|y_0\| + \sum_{r=i_0}^{i-1} \sum_{j=i_0}^{r} \|B(r,j)y(j)\| \right] \le C_1 \left[\|y_0\| + \sum_{j=i_0}^{i-1} y(j) \sum_{r=j}^{i-1} \|B(r,j)\| \right].$$

Form this and the discrete version of the Gronwall inequality, it follows that

$$\|y(i)\| \le C_1 \|y_0\| \exp \left[c_1 \sum_{j=i_0}^{i-1} \sum_{r=j}^{i-1} \|B(r,j)\| \right].$$

Thus boundedness of (2.50) is proven. ■

Now let us add one assumption which is

$$\|A(i,j)\| < M\nu^{i-j} \qquad for\ some \quad M > 0,\ \nu \in (0,1). \tag{2.52}$$

If (2.45) is uniformly asymptotically stable and satisfy (2.53) then there exist constants λ and $\gamma \in (0,1)$ such that

$$\|R(i,j)\| \le \lambda \gamma^{i-j}, \quad i \ge j, \tag{2.53}$$

where $R(i,j)$ is a fundamental matrix of (2.45).

Theorem 2.5.3 *Assume that the system (2.45) is uniformly asymptotically stable and (2.53) holds . If*

$$\sum_{i=i_0}^{\infty} \|B(i,j)\| \gamma^{-i} < \infty, \quad \lim_{j\to\infty} \sum_{i=j}^{\infty} \|B(i,j)\| \gamma^{i-j} = 0, \tag{2.54}$$

where $\gamma \in (0,1)$*, then (2.50) is bounded.*

Proof. Using (2.52) and (2.54) in (2.50) we get

$$\begin{aligned} \|y(i)\| &\leq \lambda\gamma^{i-i_0} \|y_0\| + \lambda \sum_{r=i_0}^{i-1} \gamma^{i-r-1} \sum_{j=i_0}^{r} \|B(r,j)y(j)\| \\ &\leq \lambda\gamma^{i-i_0} \|y_0\| + \lambda \sum_{j=i_0}^{i-1} \|y_j\| \sum_{r=j}^{i-1} \|B(r,j)\| \gamma^{i-r-1}. \end{aligned} \tag{2.55}$$

Let denote $q(i) = \gamma^{-i} \|y(i)\|$ and choose some positive δ such that $\gamma(1+\delta) \leq 1$. Using (2.55), take $m > i_0$ such that for all $j > m$

$$\lambda \sum_{r=j}^{i-1} \|B(r,j)\| \gamma^{j-r-1} \leq \delta, \quad i > j. \tag{2.56}$$

Now represent the double sum at the right hand side of inequality (2.56) as

$$\sum_{j=i_0}^{i-1}\sum_{r=j}^{i-1} = \sum_{j=i_0}^{m-1} \left[\sum_{r=j}^{m-1} + \sum_{r=m}^{i-1} \right] + \sum_{j=m}^{i-1}\sum_{r=j}^{i-1}.$$

Then by virtue of (2.56)

$$\begin{aligned} q(i) \;\; &\leq \;\; \lambda\gamma^{-i_0} \|y_0\| + \lambda \left[\sum_{j=i_0}^{m-1} q(i) \sum_{r=j}^{i-1} \|B(r,j)\| \gamma^{j-r-1} \right] \\ &+ \sum_{j=i_0}^{m-1} q(j) \sum_{r=m}^{m-1} \|B(r,j)\| + \lambda \sum_{j=m}^{i-1} q(j) \sum_{r=j}^{i-1} \|B(r,j)\| \gamma^{j-r-1}. \end{aligned} \tag{2.57}$$

Note that all the values $q(j)$ are bounded for $j = i_0, ..., m$. Because of this and (2.55) the expression in square brackets at the right hand side of (2.58) is bounded. From this fact and (2.58), and (2.57), the existence of a constant C follows such that

$$q(i) \leq C + \delta \sum_{j=m}^{i-1} q(j). \tag{2.58}$$

Inequality (2.59) leads to the estimate $q(i) \leq C + (1 + \delta)^{i-m}$. Because of this estimate we get

$$\|y(i)\| \leq C \left[\gamma(1 + \delta)\right]^{i-m}.$$

The latter estimate means the boundedness of (2.50) because $\gamma(1+\delta) \leq 1$ and this completes the proof of the theorem. ■

Theorem 2.5.4 *Assume that the system has the form*

$$y(i + 1) = \sum_{j=i_0}^{i} A(i, j)y(j) + b(i), \qquad i \geq i_0, \; y(i) \in R^n, \tag{2.59}$$

with (2.45) uniformly bounded, and

$$\sum_{j=i_0}^{\infty} \|b(j)\| < \infty. \tag{2.60}$$

Then (2.60) is bounded. Moreover, if (2.45) is uniformly asymptotically stable and inequalities (2.53) and (2.61) are valid, then for any solution of system (2.60) we have

$$\lim_{i \to \infty} y(i) = 0. \tag{2.61}$$

Proof. The proof of the boundedness is similar to Theorem 2.5.2. To show (2.62), let us take any $\epsilon > 0$. Because system (2.45) is uniformly asymptotically stable, inequality (2.54) is fulfilled. Now take and fix any N such that

$$\lambda \sum_{j=N}^{\infty} \|b(j)\| < \epsilon.$$

Further, similar to (2.56) using representation (2.52) we get

$$\|y(i)\| \le \lambda \gamma^{i-i_0} \|y_0\| + \lambda \sum_{j=i_0}^{N} \gamma^{i-j-1} \|b(j)\| + \lambda \sum_{j=N}^{\infty} \|b(j)\| .$$

So, for all sufficiently large i,$|y(i)| \le \epsilon$, for arbitrary $\epsilon > 0$. ■

Consider equation (2.20) in the nonconvolution form, which is

$$y(n+1) = A(n)\, y(n) + \sum_{r=0}^{n} B(n,r)\, y(r) + g(n).$$

The result on the boundedness of these equations are based on growth properties of the resolvent matrix of a linear Volterra difference equations.

First Let us introduce these assumptions:

(i) Assume that (2.54) holds for a constant $\gamma > 1$ and $\alpha F(n,u) \le F(n, \alpha u)$, with $0 < \alpha < 1$, for all $(n,x) \in Z^+ \times R^+$;

(ii) Assume that (2.54) holds for a constant γ, $0 < \gamma < 1$ and $\beta F(n,u) \le F(n, \beta u)$, $\beta > 1$ for all $(n,u) \in Z^+ \times R^+$;

(iii) Assume that (2.54) holds for a constant $\gamma = 1$.

Then we have the following theorem.

Theorem 2.5.5 *Assume that* $\|g(n,x)\| \le F(n, \|x\|)$, *for all* $(n,x) \in Z^+ \times R^q$, *where* $F : Z^+ \times R \to R^+$ *is a monotone nondecreasing continuous function with respect to the second variable for each fixed* $n \in Z^+$. *Furthermore, suppose that one of the* *assumptions is satisfied. Then, all solution* $y(n)$ *of (2.20) such that*

$$\|y(n_0)\| < \frac{\gamma^{n_0}}{\lambda} z(n_0)$$

satisfies

$$\|y(n)\| < \gamma^n \; z(n)$$

for all $n \in Z^+$, *where* $z(n)$ *is a solution of the difference equation*

$$z(n) = z(n_0) + \sum_{j=n_0}^{n-1} \frac{\lambda}{\gamma} F(j, z(j)), \qquad z(n_0) = z_0.$$

Proof. Let us assume condition (i) holds. From the Variation of Constants Formula for (2.20), we have

$$y(n) = R(n, n_0)y_0 + \sum_{j=n_0}^{n-1} R(n, j+1)g(j, y(j)), \quad n \in Z^+.$$

Thus, from the first line of the theorem, it follows that

$$\|y(n)\| \leq \|R(n, n_0)\| \, \|y(n_0)\| + \sum_{j=n_0}^{n-1} \|R(n, j+1)\| \, \|g(j, y(j))\|$$

$$\leq \lambda \gamma^{n-n_0} \, \|y(n_0)\| + \sum_{j=n_0}^{n-1} \lambda \gamma^{n-j-1} . F(j, \|y(n)\|).$$

Hence

$$\|y(n)\| \, \gamma^{-n} \leq \lambda \, \|y(n_0)\| \, \gamma^{-n_0} + \sum_{j=n_0}^{n-1} \frac{\lambda .}{\gamma} F(j, \|y(n)\| \, \gamma^{-j}).$$

By the assumption $\lambda \|y(n_0)\| < z(n_0)\gamma^{n_0}$, we infer that

$$\|y(n)\| \gamma^{-n} - \sum_{j=n_0}^{n-1} \frac{\lambda}{\gamma} F(j, \|y(n)\| \gamma^{-j}) < z(n) - \sum_{j=n_0}^{n-1} \frac{\lambda}{\gamma} F(j, z(n)),$$

from which, it follows that

$$\|y(n)\| < \ z(n)\gamma^{n}, \quad n \in Z^{+},$$

because $\|y(n_0)\| \gamma^{-n_0} < z(n_0)$ and $F(r,u)$ is a monotone nondecreasing continuous function with respect to u for each $r \in Z^{+}$.

In the case that (ii) or (iii) holds, the proof is carried out similarly, which completes the proof of the theorem. ■

2.6 Volterra Difference Equations with Degenerate Kernels

In this section, new form will be discussed which is the implicit form of Volterra difference equations with degenerate kernels. Stability criteria are derived. The main tool to find the stability criteria in this analysis is the use of the new representation formula which allows us to express the solution of Volterra difference equations with degenerate kernels in terms of the fundamental matrix of the corresponding first-order system of the difference equations. First, we will introduce definitions of the degenerate kernels.

Definition 2.6.1 *The kernels k is said to be degenerate or of Pincherle-Gourast type (PG kernel) if there exist continuous $q \times q-$ matrix functions $A(i,t)$ and $B(i,s)$, $i = 1, \; 2, \; \ldots, \; p,$ such that*

$$k(t,s) = \sum_{i=1}^{p} A(i,t)B(i,s). \tag{2.62}$$

Consider the equation

$$y(n) = g(n) + \sum_{j=n_0}^{n} k(n,j)y(j), \qquad n \geq n_0, \tag{2.63}$$

$y(n)$ and $g(n) \in R^q$, $k(n,j) \in R^{q\times q}$, where the kernel $k(n,j)$ is of PG-type, i.e.,

$$k(n,j) = \sum_{i=1}^{p} A(i,n)B(i,j), \tag{2.64}$$

for some matrix sequences $A(i,n)$ and $B(i,j) \in R^{q\times q}$.

Consider the matrix equation

$$u(m,n) = I + \sum_{j=m}^{n} k(n,j)u(m,j). \tag{2.65}$$

Here $u(m,n)$ is a $q \times q$ matrix which is the fundamental matrix for (2.64). Observe that $u(n+1,n) = I$. This equation has a solution if, $\|k(n,j)\| < 1$. It can be checked by direct substitution that the solution $y(n)$ to (2.64) with $n_0 = 0$ can be written as

$$y(n) = g(n) - \sum_{j=0}^{n} (u(j+1,n) - u(j,n))g(j), \qquad n \geq 0. \tag{2.66}$$

Consider now (2.64) with the kernel of the form (2.65), i.e.,

$$y(n) = g(n) + \sum_{v=1}^{p} A(v,n) \sum_{j=1}^{n} B(v,j) y(j), \qquad n \geq 0. \tag{2.67}$$

Putting

$$z(v,n) = \sum_{j=1}^{n} B(v,j) y(j),$$

the solution to (2.68) can be expressed as

$$y(n) = g(n) + \sum_{j=1}^{p} A(j,n) z(j,n), \tag{2.68}$$

where $z(j,n)$ is the solution to the system

$$z(v,n+1) = z(v,n) + B(v,n+1) \sum_{j=1}^{p} A(j,n+1) z(j,n+1) + B(v,n+1) g(n+1). \tag{2.69}$$

Define the column vector $Z(n)$ of length pq and the $pq \times q$ matrices $A(n)$ and $B(n)$ by

$$Z(n) = \begin{bmatrix} Z(1,n) \\ Z(2,n) \\ \vdots \\ Z(p,n) \end{bmatrix}, \quad A(n) = \begin{bmatrix} A(1,n) \\ A(2,n) \\ \vdots \\ A(p,n) \end{bmatrix}, \quad \text{and } B(n) = \begin{bmatrix} B(1,n) \\ B(2,n) \\ \vdots \\ B(p,n) \end{bmatrix},$$

and define the $pq \times pq$ matrix $M(n)$ by

$$M(n) = \left[B(i,n) A(j,n) \right]_{i,j=1}^{p}.$$

Then system (2.70) can be written in the matrix form

$$(I - M(n+1))Z(n+1) = Z(n) + B(n+1)g(n+1).$$

Consider also the homogeneous system

$$(I - M(n))Y(n+1) = Y(n), \tag{2.70}$$

where the matrix $M(n+1)$ has been replaced by $M(n)$, and denote $Y(n,j)$ the fundamental matrix of (2.71); i.e., the matrix such that

$$(I - M(n))Y(n+1,j) = Y(n,j), \qquad Y(j,j) = I.$$

Such a matrix exists if, for example, $\|M(n)\| < 1$, which is the condition imposed in stability criteria presented in the coming theorems. Let

$$Y(i,j) = [Y(v,j;\mu,i)]^{p}_{v,\ \mu=1}\,.$$

We have the following theorem.

Theorem 2.6.1 *Assume that the fundamental matrix $Y(n,j)$ to (2.71) exists. Then the solution $u(m,n)$ to (2.66) with kernel $k(n,j)$ of the form (2.65) is given by*

$$u(m,n) = I + \sum_{\mu=1}^{p}\sum_{v=1}^{p}\sum_{i=m}^{n} A(v,n)Y(v,n+1;\mu,i)B(\mu,i) \tag{2.71}$$

Proof. It follows that

$$\begin{aligned}
E &= I+\sum_{j=m}^{n}k(n,j)u(m,j)\\
&= I+\sum_{j=m}^{n}\sum_{\eta=1}^{p}A(\eta,n)B(n,j)\left[I+\sum_{\mu=1}^{p}\sum_{v=1}^{p}\sum_{i=m}^{j}A(v,j)Y(v,j+1;\mu,i)B(\mu,i)\right]\\
&= I+\sum_{j=m}^{n}\sum_{\eta=1}^{n}A(\eta,n)B(n,j)\\
&\quad +\sum_{j=m}^{n}\sum_{\eta=1}^{p}\sum_{v=1}^{p}\sum_{\mu=1}^{p}\sum_{i=m}^{j}A(\eta,n)B(\eta,j)A(v,j)Y(v,j+1;\mu,i)B(\mu,i).
\end{aligned}$$

Observing that $\sum\limits_{j=m}^{n}\sum\limits_{i=m}^{j}=\sum\limits_{i=m}^{n}\sum\limits_{j=i}^{n}$ we obtain

$$\begin{aligned}
E &= I+\sum_{j=m}^{n}\sum_{\eta=1}^{n}A(\eta,n)B(n,j)\\
&\quad +\sum_{\eta=1}^{p}\sum_{\mu=1}^{p}\sum_{i=m}^{n}A(\eta,n)\left(\sum_{j=i}^{n}\sum_{v=1}^{p}B(\eta,j)A(v,j)Y(v,j+1;\mu,i)\right)B(\mu,i).
\end{aligned}$$

Since $Y(i,j)$ is the fundamental matrix of (2.71) it follows that

$$Y(\eta,j+1;\mu,i)=Y(\eta,j;\mu,i)+\sum_{v=1}^{p}B(\eta,j)A(v,j)Y(v,j+1;\mu,i).$$

Substituting this relation into the expression for E we get

$$\begin{aligned}
E &= I + \sum_{j=m}^{n}\sum_{\eta=1}^{n} A(\eta,n)B(n,j) \\
&\quad + \sum_{\eta=1}^{p}\sum_{\mu=1}^{p}\sum_{i=m}^{n} A(\eta,n)\sum_{j=i}^{n}\left(Y(\eta,j+1;\mu,i) - Y(\eta,j;\mu,i)\right)B(\mu,i) \\
&= I + \sum_{j=m}^{n}\sum_{\eta=1}^{n} A(\eta,n)B(n,j) \\
&\quad + \sum_{\eta=1}^{p}\sum_{\mu=1}^{p}\sum_{i=m}^{n} A(\eta,n)\sum_{j=i}^{n}\left(Y(\eta,j+1;\mu,i) - \delta(\eta,\mu)\right)B(\mu,i) \\
&= I + \sum_{\eta=1}^{p}\sum_{\mu=1}^{p}\sum_{i=m}^{n} A(\eta,n)\sum_{j=i}^{n} Y(\eta,j+1;\mu,i)B(\mu,i) \\
&= u(m,n),
\end{aligned}$$

where $\delta(\eta,\mu)=1$ if $\eta=\mu$ and $\delta(\eta,\mu)=0$ otherwise. This completes the proof. ■

Formula (2.72) is used in the coming parts to derive various stability criteria for (2.64) with PG kernels.

In what follows we assume without loss of generality that $n_0 = 0$. We have the following criteria for uniform stability of (2.64) with PG-kernel.

Theorem 2.6.2 *Suppose that the kernel $k(n,j)$ is PG-kernel. Assume that $\|M(n)\| < 1,\ n \geq 0$, and that there exists a constant D such that*

$$\sum_{\mu=1}^{p}\sum_{v=1}^{p}\sum_{j=0}^{n}\left(\frac{\|A(v,n)\|\,\|B(\mu,j)\|}{\prod_{l=j}^{n}\left(1-\|M(l)\|\right)}\right) \leq D,$$

for $n \geq 0$, where $M(n) = [B(i,l)A(j,l)]_{i,j=1}^{p}$. Then (2.64) is uniformly stable.

Proof. It follows form (2.67) that

$$\|y(n)\| \leq \|g(n)\| + \sum_{j=0}^{n} \|u(j+1,n) - u(j,n)\| \, \|g(j)\| \, .$$

Using the representation formula (2.72) we get

$$u(j+1,n) - u(j,n) = -\sum_{\mu=1}^{p}\sum_{v=1}^{p} A(v,n)Y(v,n+1;\mu,j)B(\mu,j),$$

and

$$\begin{aligned}
\sum_{j=0}^{n} \|u(j+1,n) - u(j,n)\| \quad &\leq \quad \sum_{\mu=1}^{p}\sum_{v=1}^{p}\sum_{j=0}^{n} \|A(v,n)\| \, \|Y(v,n+1;\mu,j)\| \, \|B(\mu,j)\| \\
&\leq \quad \sum_{\mu=1}^{p}\sum_{v=1}^{p}\sum_{j=0}^{n} \|A(v,n)\| \, \|Y(n+1,j)\| \, \|B(\mu,j)\| \, .
\end{aligned}$$

Since $\|M(n)\| < 1$ the matrix $I - M(n)$ is nonsingular and the fundamental matrix $Y(n+1,j)$ of the system (2.71) can be written as

$$Y(n+1,j) = \prod_{l=j}^{n} (I - M(n+j-l))^{-1}.$$

Hence

$$\|Y(n+1,j)\| \leq \frac{1}{\prod\limits_{l=j}^{n} (1 - \|M(l)\|)},$$

and

$$\sum_{j=0}^{n} \|u(j+1,n) - u(j,n)\| \leq \sum_{\mu=1}^{p}\sum_{v=1}^{p}\sum_{j=0}^{n} \left(\frac{\|A(v,n)\| \, \|B(\mu,j)\|}{\prod\limits_{l=j}^{n} (1 - \|M(l)\|)} \right).$$

Substituting this relation into the inequality for $\|y(n)\|$ we obtain

$$\|y(n)\| \leq (1+D)\,\|g(n)\|\,,$$

which completes the proof. ■

To illustrate this theorem, consider (2.64) with $p = q = 1$ and the kernel $k(n,j) = a(n)b(j)$, where $a(n)$ and $b(j)$ are defined by

$$a(n) = \frac{1}{(n+1)(n+2)}, \qquad b(j) = j+1.$$

We have $M(l) = a(l)b(l) = \frac{1}{l+2}$ and it follows that

$$\sum_{j=0}^{n}\left(\frac{a(n)b(j)}{\prod\limits_{l=j}^{n}(1-a(l)b(l))}\right) = \frac{1}{(n+1)(n+2)}\sum_{j=0}^{n}\left(\frac{(j+1)}{\prod\limits_{l=j}^{n}\frac{l+1}{l+2}}\right) = 1.$$

Hence, in view of this theorem, equation (2.64) is uniformly stable.

Theorem 2.6.3 *Suppose that the kernel $k(n,j)$ is PG kernel. Assume that $\|M(n)\| < 1$, $n \geq 0$, $\|g(n)\| \to 0$ as $n \to \infty$, and that*

$$\sum_{\mu=1}^{p}\sum_{v=1}^{p}\sum_{j=0}^{n}\left(\frac{\|A(v,n)\|\,\|B(\mu,j)\|}{\prod\limits_{l=j}^{n}(1-\|M(l)\|)}\right) \to 0 \qquad \text{as } n \to \infty.$$

Then (2.64) is asymptotically stable.

Proof. Proceeding similarly as in the proof of Theorem 2.6.2 we obtain

$$\sum_{\mu=1}^{p}\sum_{v=1}^{p}\sum_{j=0}^{n}\left(\frac{\|A(v,n)\|\,\|B(\mu,j)\|}{\prod_{l=j}^{n}(1-\|M(l)\|)}\right)\to 0 \qquad \text{as } n\to\infty,$$

and completes the proof of the theorem. ■

2.7 Nonlinear Volterra Difference Equations

We study the boundedness of solutions and stability properties of the zero solution of the nonlinear perturbed Volterra discrete system

$$x(n+1) = A(n)\,x(n) + \sum_{s=0}^{n} B(n,s)f(r,\,x(s)) + g(n,x(n)), \tag{2.72}$$

where $g(n,x(n))$ and $f(n,x(n))$ are $k\times 1$ vector functions that are continuous in x and satisfy

$$|g(n,x(n))| \leq \lambda_1(n) + \lambda_2(n)\,|x(n)|\,,$$

and

$$|f(n,x(n))| \leq \gamma(n)\,|x(n)|\,.$$

Assuming that $\gamma(n)$ is positive and bounded, $0\leq\lambda_1(n)\leq M$ and $0\leq\lambda_2(n)\leq L$ for some positive constants M and L. Moreover, $A(n)$ and $B(n,r)$ are $k\times k$ matrix functions on Z^+ and $Z^+\times Z^+$. For $x\in R$, $\|x\| = \max\limits_{1\leq i\leq k}|x_i|$ and if $A=(a_{ij})$ is $k\times k$ real matrix, then

$$\|A\| = \max_{1\leq i\leq k}\sum_{j=1}^{k}|a_{ij}|\,.$$

We say that $x(n) = x(n, n_0, \phi)$ is a solution of (2.73) with a bounded initial function $\phi : [0, n_0] \to R^k$ if it satisfies (2.73) for $n > n_0$ and $x(j) = \phi(j)$ for $j \leq n_0$. Let us rewrite some definitions with our notations that are needed.

Definition 2.7.1 *Solutions of (2.73) are uniformly bounded if for each $B_1 > 0$ there is $B_2 > 0$ such that if $n_0 \geq 0$, $\phi : [0, n_0] \to R^k$ with $\|\phi(n)\| < B_1$ on $[0, n_0]$, implies*

$$|x(n, n_0, \phi)| < B_2,$$

for $n \geq n_0$ where $\|\phi\| = \sup |\phi(n)|\,, \ \ 0 \leq n \leq n_0$.

Definition 2.7.2 *The zero solution of (2.73) is stable if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon)$ such that if $\phi : [0, n_0] \to R^k$ with $\|\phi(n)\| < \delta$ on $[0, n_0]$, implies*

$$|x(n, n_0, \phi)| < \varepsilon.$$

It is uniformly stable if δ is independent of n_0.

Definition 2.7.3 *The zero solution of (2.73) is uniformly asymptotically stable if it is uniformly stable and there is $\eta > 0$ such that for each $\mu > 0$ there exists $N(\mu) > 0$ independent of n_0, such that*

$$|x(n, n_0, \phi)| < \mu,$$

for all $n \geq n_0 + N(\mu)$, whenever $\|\phi(n)\| < \eta$ on $[0, n_0]$.

We will be using Liapunov functionals to prove the following theorem.

Theorem 2.7.1 *Suppose there is a function $\phi(n) \geq 0$ with $\triangle\phi(n) \leq 0$ for $n \geq 0$,*

$$\sum_{n=0}^{\infty} |\phi(n)| < \infty,$$

and

$$\triangle_n \phi(n-s-1) + |B(n,s)|\,\gamma(s) \leq 0,$$

for $0 \leq s < n < \infty$. If for $n \geq 0$,

$$|A(n)| + |B(n,n)|\,\gamma(n) + \lambda_2(n) + \phi(0) \leq 1 - \alpha$$

for some $\alpha \in (0,1)$, then all solutions of (2.73) are uniformly bounded. Moreover, if $\lambda_1(n) = 0$ then the zero solution of (2.73) is uniformly asymptotically stable.

Proof. Define

$$V(n, x(\cdot)) = |x(n)| + \sum_{s=0}^{n-1} \phi(n-s-1)\,|x(s)|\,. \tag{2.73}$$

Along solutions of (2.73), we have

$$\begin{aligned}
\triangle V(n, x(\cdot)) &= |x(n+1)| - |x(n)| + \sum_{s=0}^{n} \phi(n-s)\,|x(s)| - \sum_{s=0}^{n-1} \phi(n-s-1)\,|x(s)| \\
&\leq |A(n)|\,|x(n)| + \sum_{s=0}^{n} |B(n,s)|\,\gamma(s)\,|x(s)| + \lambda_2(n)\,|x(n)| \\
&\quad + \lambda_1(n) + \sum_{s=0}^{n} \phi(n-s)\,|x(s)| - \sum_{s=0}^{n-1} \phi(n-s-1)\,|x(s)| - |x(n)| \\
&= |A(n)|\,|x(n)| + \sum_{s=0}^{n-1} |B(n,s)|\,\gamma(s)\,|x(s)| + |B(n,n)|\,\gamma(n)\,|x(n)| \\
&\quad + \lambda_1(n) + \lambda_2(n)\,|x(n)| + \phi(0)\,|x(n)| \\
&\quad + \sum_{s=0}^{n-1} \phi(n-s)\,|x(s)| - \sum_{s=0}^{n-1} \phi(n-s-1)\,|x(s)| - |x(n)| \\
&= \left[|A(n)| + |B(n,n)|\,\gamma(n) + \lambda_2(n) + \phi(0) - 1\right]|x(n)| + \lambda_1(n) \\
&\quad + \sum_{s=0}^{n-1} \left[|B(n,s)|\,\gamma(s) + \triangle_n \phi(n-s-1)\right]|x(s)| \\
&\leq -\alpha\,|x(n)| + \lambda_1(n) \\
&\leq -\alpha\,|x(n)| + M.
\end{aligned}$$

Next, we assume that $\lambda_1(n) = 0$ and show that the zero solution of (2.73) is uniformly asymptotically stable. From (2.74), we have that

$$\triangle V(n, x(\cdot)) \leq -\alpha\,|x(n)|\,.$$

Let $\rho = 1 + \sum_{s=0}^{\infty} \phi(s)$. As

$$\triangle V(n, x(\cdot)) \leq 0,$$

it follows that

$$\begin{aligned}
|x(n,n_0,\phi)| &\leq \triangle V(n,x(\cdot)) \leq V(n_0,\phi) \\
&\leq |\phi| + \sum_{s=0}^{n_0-1} \phi(n_0-s-1)\,|\phi| \\
&\leq \|\phi\| \left[1 + \sum_{s=0}^{\infty} \phi(s)\right] \\
&\leq \varepsilon,
\end{aligned}$$

for $\|\phi\| \leq \delta$ with $\delta = \frac{\varepsilon}{\rho}$. Hence the zero solution of (2.73) is uniformly stable. For the rest, we follow the proof of Theorem 2.4.1. This completes the proof. ■

Chapter 3

Discretization of Volterra Equations

In this chapter we will establish the discrete form of the Volterra integral equations and the Volterra integro-differential equations by the generation of quadrature weights of Backward Euler, Trapezoidal, and Simpson & Trapezoidal method and the Linear Multistep methods. For existence, uniqueness, and stability properties of these methods see [3], [5], [11], [20], [37], [39], [40].

3.1 Discretization of Volterra Integral and Integro-differential Equations

Let us consider the scalar linear Volterra integral equation

$$y(t) = g(t) + \int_{t_0}^{t} k(t,s)\, y(s) ds, \qquad t \in [t_0, T], \qquad y, g, k \in R. \tag{3.1}$$

The application of a Direct Quadrature method to (3.1) leads to

$$y_n = g_n + h \sum_{l=0}^{n} w_{n,l}\, k_{n,l}\, y_l, \qquad n \geq n_0, \tag{3.2}$$

given $y_0, y_1, ..., y_{n_0-1}$, given, where $t_n = t_0 + nh, \quad h = (T - t_0)/N, \quad k_{n,l} = k(t_n, t_l)$, $g_n = g(t_n), \quad y_n = y(t_n)$ and $w_{n,l}$ are given coefficients. These quadrature coefficients are obtained from numerical methods such as Backward Euler, Trapezoidal, and Simpson & Trapezoidal see [4], [11], [15], [17], [19], [23], [33]. The weights can be written in the form of infinite matrices

$$W = \ \{w_{n,l}\}\,, \quad n \geq n_0, \quad n_0 \geq l \geq n\ .$$

Backward Euler (BE) method, order 1 and n_0 =1

$$W = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.3}$$

Trapezoidal (TR) method, order 2 and n_0 =1

$$W = \begin{pmatrix} \frac{1}{2} & & & & \\ 1 & \frac{1}{2} & & & \\ 1 & 1 & \frac{1}{2} & & \\ 1 & 1 & 1 & \frac{1}{2} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.4}$$

Simpson +Trapezoidal (ST) method, order 3 and n_0 =2

$$W = \begin{pmatrix} \frac{1}{3} & & & & & & \\ \frac{1}{3}+\frac{1}{2} & \frac{1}{2} & & & & & \\ \frac{1}{3}+\frac{1}{3} & \frac{4}{3} & \frac{1}{3} & & & & \\ \frac{1}{3}+\frac{1}{3} & \frac{4}{3} & \frac{1}{3}+\frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{3}+\frac{1}{3} & \frac{4}{3} & \frac{1}{3}+\frac{1}{3} & \frac{4}{3} & \frac{1}{3} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}. \tag{3.5}$$

Now we investigate the existence and uniqueness of the solution of Volterra integral equations.

3.1.1 Existence and Uniqueness of the Solution

The classical approach to proving the existence and uniqueness of the solution of (3.1) is the method of successive approximation, also called the Picard method, which consists of the simple iteration

$$y_n(t) = g(t) + \int_0^t k(t,s)\; y_{n-1}(s)ds, \qquad n = 1, 2, \dots, \tag{3.6}$$

with

$$y_0(t) = g(t).$$

Let us introduce

$$\varphi_n(t) = y_n(t) - y_{n-1}(t), \qquad n = 1, 2, \dots, \tag{3.7}$$

with

$$\varphi_0(t) = g(t).$$

On subtracting from (3.6) the same equation with n replaced by $n-1$, we obtain

$$\varphi_n(t) = \int_0^t k(t,s)\ \varphi_{n-1}(s)ds, \qquad n = 1, 2, \ldots. \tag{3.8}$$

Also, from (3.7)

$$y_n(t) = \sum_{i=0}^{n} \varphi_i(t). \tag{3.9}$$

Now we will use this iteration to prove the existence and uniqueness of the solution.

Theorem 3.1.1 *If* $k(t,s)$ *is continuous in* $0 \le s \le t \le T$ *and* $g(t)$ *is continuous in* $0 \le t \le T$, *then (3.1) has a unique continuous solution for* $0 \le t \le T$.

Proof. Choose G and K such that

$$\begin{aligned} |g(t)| &\le G, \qquad 0 \le t \le T, \\ |k(t,s)| &\le K, \qquad 0 \le s \le t \le T. \end{aligned}$$

We first prove by induction that

$$|\varphi_n(t)| \le \frac{G(Kt)^n}{n!}, \qquad 0 \le t \le T, \quad n = 0, 1, \ldots. \tag{3.10}$$

If we assume that (3.10) is true for $n-1$, then from (3.8)

$$|\varphi_n(t)| \le \frac{G(Kt)^n}{(n-1)!} \int_0^t s^{n-1} ds \ \le \frac{G(Kt)^n}{n!}.$$

Since (3.10) is obviously true for $n = 0$, it holds for all n. This bound makes it obvious that the sequence $y_n(t)$ in (3.9) converges and we can write

$$y(t) = \sum_{i=0}^{\infty} \varphi_i(t). \tag{3.11}$$

We now show that this $y(t)$ satisfies (3.1).

The series (3.11) is uniformly convergent since the terms $\varphi_i(t)$ are dominated by $\frac{G(KT)^i}{i!}$. Hence we can interchange the order of integration and summation in the following expression to obtain

$$\begin{aligned}
\int_0^t k(t,s) \sum_{i=0}^{\infty} \varphi_i(s) ds &= \sum_{i=0}^{\infty} \int_0^t k(t,s) \varphi_i(s) ds \\
&= \sum_{i=0}^{\infty} \varphi_{i+1}(t) = \sum_{i=0}^{\infty} \varphi_i(t) - g(t).
\end{aligned}$$

This proves that $y(t)$ defined by (3.11) satisfies (3.1). Each of $\varphi_i(t)$ is clearly continuous. Therefore $y(t)$ is continuous, since it is the limit of a uniformly convergent sequence of continuous functions.

To show that $y(t)$ is the only continuous solution, suppose there exists another continuous solution $\hat{y}(t)$ of (3.1). Then

$$y(t) - \hat{y}(t) = \int_0^t k(t,s) \, (y(s) - \hat{y}(s)) ds. \tag{3.12}$$

Since $y(t)$ and $\hat{y}(t)$ are both continuous, there exists a constant B such that

$$|y(t) - \hat{y}(t)| \leq B, \qquad 0 \leq t \leq T.$$

Substituting this into (3.12)

$$|y(t) - \hat{y}(t)| \leq KBt, \qquad 0 \leq t \leq T,$$

and repeating the step shows that

$$|y(t) - \hat{y}(t)| \leq \frac{B(Kt)^n}{n!}, \qquad 0 \leq t \leq T,$$

for any n. For large enough n , the right-hand side is arbitrarily small, so that we must have

$$y(t) = \hat{y}(t), \qquad 0 \leq t \leq T,$$

and there is only one continuous solution. ■

3.1.2 Boundedness of the Direct Quadrature Method

As we know, because of the linearity of the problem (3.1), the global error

$$e_n = y(t_n) - y_n$$

of the method satisfies an analogous Volterra difference equations

$$e_n = T_n(h) + h \sum_{l=0}^{n} w_{n,l}\, k_{n,l}\, e_l, \quad n \geq n_0, \tag{3.13}$$

where $T_n(h)$ represents the local discretization error see [18] and it is assumed that

$$T_n(h) < T^*, \quad n \geq n_0. \tag{3.14}$$

We will introduce the stability of the Direct Quadrature Method in the sense of boundedness of global error. In order to show this, we first rewrite (3.13) in the following form

$$e_n = \hat{g}_n + \sum_{l=n_0}^{n} a_{n,l}\, e_l, \quad n \geq n_0, \tag{3.15}$$

with

$$\hat{g}_n = T_n(h) + \sum_{l=0}^{n_0-1} w_{n,l}\, k_{n,l}\, e_l, \qquad a_{n,l} = hw_{n,l}\, k_{n,l}.$$

Now by assuming that $e_0, e_1, ..., e_{n_0-1}$ are given and recalling (3.15), we have the result that the global error satisfies a Volterra difference equations of the form (2.4) with

$$|\ \hat{g}_n| < \ \hat{g}, \qquad n \geq n_0. \tag{3.16}$$

Theorem 3.1.2 *Assume that*

i) $k(t,s) \leq 0, \qquad t \geq t_0, \qquad t_0 \leq s \leq t,$

ii) $\frac{\partial k(t,s)}{\partial t} \geq 0, \qquad t \geq t_0, \qquad t_0 \leq s \leq t,$

iii) $|k(t,s)| \leq \phi(t),$

iv) $\phi'(t) \leq 0,$

v) $\int\limits_{t_0}^{\infty} \phi(t)dt = k^* < \infty,$

vi) $h\phi(t_0) \leq c_1.$

Then the global error of the Direct Quadrature Method under consideration satisfies

$$|e_n| \leq \hat{g}(1 + c_2 k^*).$$

Proof. We give only the proof for the case of the TR Method since remaining methods can be handled analogously. The Volterra difference equation (3.15) representing the global error of the Trapezoidal Method satisfies

$$\begin{aligned} x_n &= g_n + \sum_{l=n_0}^{n} a_{n,l}\, x_l, \qquad n \geq n_0, \\ a_{n.n} &\neq 1, \quad n \geq n_0, \end{aligned}$$

with

$$a_{n,l} = \begin{cases} hk_{n,l}, & 1 \leq l \leq n-1 \\ \frac{h}{2}k_{n,n}, & l = n \end{cases}, \qquad n \geq 1$$

and hence $a_{n+1,l} \geq a_{n,l}, \quad 1 \leq l \leq n-1$. From (vi) and taking into account (iii) and (i) one finds

$$\frac{h}{2}\,|k_{n,n}| < 1,$$

and thus

$$hk_{n+1,n} - \frac{h}{2}k_{n,n} \geq \frac{h}{2}k_{n,n} > -1, \quad n \geq 1.$$

This assures that

$$a_{n+1,n} - a_{n,n} \geq -1, \quad n \geq 1.$$

Then by the application of Theorem 2.2 in [39] and taking into account that

$$\sum_{l=1}^{\infty} |a_{l+1,l}| \leq h\sum_{l=1}^{\infty} |k_{l,l}| \leq \int_{t_0}^{\infty} \phi(t)dt = k^*,$$

we have that the fundamental matrix of (3.15) satisfies

$$\sum_{m=1}^{n-1} |u_{m+1,n} - u_{m,n}| \leq 2k^*.$$

Finally the desired result follows from (3.16) and

$$x_n = g_n + \sum_{l=n_0}^{n} (u_{l+1,n} - u_{l,n})\, g_l, \qquad n \geq n_0.$$

This completes the proof of the theorem. ■

Specific program to solve this Volterra integral equations is presented in Appendix A.

Example 3.1.1 *Consider Volterra integral equations*

$$y\ (t) = \frac{1}{2}t^2 \exp(-t) + \frac{1}{2}\int_0^t (t-s)^2 \exp(s-t) y(s)ds, \qquad y(0) = 0, \qquad 0 \leq t \leq 6,$$

which has exact solution

$$y(t) = \frac{1}{3} - \frac{1}{3}\exp(-\frac{3t}{2})\left\{\cos(\frac{1}{2}t\sqrt{3}) + \sqrt{3}\sin(\frac{1}{2}t\sqrt{3})\right\}.$$

Using (3.2) and the program in Appendix A we get the results shown in Table 3.1

t	h	exact	BE y(t)	error	TR y(t)	error	ST y(t)	error
6	$\frac{1}{10}$	0.3334	0.3334	5.0876×10^{-7}	0.3334	5.0876×10^{-7}	0.3334	2.2263×10^{-5}
6	$\frac{1}{20}$	0.3334	0.3334	3.1820×10^{-8}	0.3334	3.1820×10^{-8}	0.3334	2.9286×10^{-6}
6	$\frac{1}{30}$	0.3334	0.3334	6.2862×10^{-9}	0.3334	6.2862×10^{-9}	0.3334	8.8283×10^{-7}
6	$\frac{1}{40}$	0.3334	0.3334	1.989×10^{-9}	0.3334	1.9890×10^{-9}	0.3334	3.7568×10^{-7}
6	$\frac{1}{50}$	0.3334	0.3334	8.1474×10^{-10}	0.3334	8.1474×10^{-10}	0.3334	1.9335×10^{-7}
6	$\frac{1}{60}$	0.3334	0.3334	3.9292×10^{-10}	0.3334	3.9292×10^{-10}	0.3334	1.1228×10^{-7}
6	$\frac{1}{70}$	0.3334	0.3334	2.1209×10^{-10}	0.3334	2.1209×10^{-10}	0.3334	7.0886×10^{-8}

Table 3.1: Results of Example 3.1.1

Now, we will deal with Volterra integro-differential equations similar to (2.2)

$$\acute{x}(t) = g(t) + \int_0^t k(t,s)x(s)ds. \tag{3.17}$$

For the development of numerical methods it is convenient to rewrite (3.17) as

$$\acute{x}(t) = H(t, x(t), z(t)), \tag{3.18}$$

$$z(t) = \int_0^t k(t,s)x(s)ds. \tag{3.19}$$

We then integrate (3.18) from t_{n-1} to t_n gives

$$x(t_n) = x(t_{n-1}) + \int_{t_{n-1}}^{t_n} H(s, x(s), z(s))ds. \tag{3.20}$$

If we apply for example the trapezoidal rule, we get

$$x_n = x_{n-1} + \frac{h}{2}\left\{H(t_{n-1}, x_{n-1}, z_{n-1}) + H(t_n, x_n, z_n)\right\}, \qquad x_0 = x(0), \tag{3.21}$$

where $z_0 = 0$, and

$$z_n = \frac{h}{2}k(t_n, t_0)x_0 + h\sum_{i=1}^{n-1}k(t_n, t_i)x_i + \frac{h}{2}k(t_n, t_n)x_n, \qquad n = 1, 2, ... \tag{3.22}$$

Using (3.21) and (3.22), the discrete form of (3.17) follows

$$\begin{aligned} x_n &= x_{n-1} + \frac{h}{2}[(g_{n-1} + g_n) + h\sum_{i=0}^{n-1}(w_{n-1,i}k_{n-1,i} + w_{n,i}k_{n,i})\,x_i \\ &\quad + hw_{n,n}k_{n,n}x_n], \end{aligned} \tag{3.23}$$

or

$$\begin{aligned} x_n &= \frac{1}{1-\frac{h^2}{2}w_{n,n}k_{n,n}}\{x_{n-1}+\frac{h}{2}[(g_{n-1}+g_n) \\ &\quad +h\sum_{i=0}^{n-1}(w_{n-1,i}k_{n-1,i}+w_{n,i}k_{n,i})\,x_i]\}, \end{aligned} \tag{3.24}$$

where $w_{n,i}$ is the quadrature weights given by (3.4) and the order of this method is two.

Higher order methods can be constructed in a similar way. For example, if we integrate (3.18) from t_{n-2} to t_n and using Simpson's rule we obtain

$$\begin{aligned} x_n &= x_{n-1}+\frac{h}{3}[(g_{n-2}+h\sum_{i=0}^{n-2}w_{n-2,i}k_{n-2,i}x_i)+4(g_{n-1}+h\sum_{i=0}^{n-1}w_{n-1,i}k_{n-1,i}x_i) \\ &\quad +(g_n+h\sum_{i=0}^{n}w_{n,i}k_{n,i}x_i)], \end{aligned} \tag{3.25}$$

rewrite (3.25) in this form

$$\begin{aligned} x_n &= A_n\,\{x_{n-2}+\frac{h}{3}[G_n+(h\sum_{i=0}^{n-2}(w_{n-2,i}k_{n-2,i}+4w_{n-1,i}k_{n-1,i}+w_{n,i}k_{n,i})x_i) \\ &\quad +4hw_{n-1,n-1}k_{n-1,n-1}x_{n-1}]\}, \end{aligned} \tag{3.26}$$

where

$$A_n = \frac{1}{1-\frac{h^2}{3}w_{n,n}k_{n,n}}, \quad G_n = g_{n-2}+4g_{n-1}+g_n,$$

and x_0, x_1 are given and $n \geq 2$. The weights $w_{n,i}$ are determined by whatever numerical integration rule we choose.

Specific programs to solve this Volterra integro-differential equations are presented in Appendices B and C.

Example 3.1.2 *Consider Volterra integro-differential equations*

$$\acute{y}\ (t) = (1+\exp(\frac{1}{8})t)+(-\frac{7}{8})\int_0^t \exp\left[(-\frac{7}{8})(s-t)\right] y(s)ds, \quad y(0)=2, \quad 0 \le t \le 10,$$

which has exact solution

$$y(t) = 1 + \exp(\frac{1}{8})t. \tag{3.27}$$

Using (3.23) and the program in Appendix B with Trapezoidal's weights, we get the results shown in Table 3.2

t	h	exact	BE+TR y(t)	error
10	$\frac{1}{10}$	4.4903	4.3182	1.7210×10^{-1}
10	$\frac{1}{20}$	4.4903	4.4153	7.5000×10^{-2}
10	$\frac{1}{30}$	4.4903	4.4484	4.2000×10^{-2}
10	$\frac{1}{40}$	4.4903	4.4650	2.5300×10^{-2}
10	$\frac{1}{50}$	4.4903	4.4750	1.5300×10^{-2}
10	$\frac{1}{75}$	4.4903	4.4891	1.2000×10^{-3}
10	$\frac{10}{785}$	4.4903	4.4903	1.2353×10^{-5}

Table 3.2: Results of Example 3.1.2 with Trapezoidal's weights

Moreover, using (3.25) and the program in Appendix C with Simpson's weights, we get the results shown in Table 3.3

t	h	exact	BE+SM y(t)	error
10	$\frac{1}{10}$	4.4903	4.1160	7.8800×10^{-2}
10	$\frac{1}{20}$	4.4903	4.4622	2.8200×10^{-2}
10	$\frac{1}{30}$	4.4903	4.4798	1.0500×10^{-2}
10	$\frac{1}{40}$	4.4903	4.4888	1.5000×10^{-3}
10	$\frac{1}{41}$	4.4903	4.4895	8.3737×10^{-4}
10	$\frac{1}{42}$	4.4903	4.4901	2.0582×10^{-4}

Table 3.3: Results of Example 3.1.2 with Simpson's weights

3.2 Discretization of Volterra Integral and Integro-differential Equations Using Linear Multistep Methods

In this part, we first describe the methods and introduce the notation that we shall use. Next, we apply this method to our equations. Now, fix a stepsize $h > 0$ and apply a Linear Multistep Method to the ordinary differential equation $\acute{y} = f(x, y)$, we get

$$\sum_{j=0}^{k} \alpha_j y_{n+j-k} = h \sum_{j=0}^{k} \beta_j f(x_{n+j-k}, y_{n+j-k}), \qquad n \geq 0, \tag{3.28}$$

with given starting values $y_{-k}, y_{-k+1}, \ldots, y_{-1}$. Here $x_n = x_0 + nh$ with $x_0 = kh$. Let

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \qquad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j, \tag{3.29}$$

denote the generating polynomials of the method. We shall make the usual assumptions on (ρ, σ)

(i) ρ , σ have no common root;

(ii) the method is consistent, i.e. $\rho(1) = 0$ and $\acute{\rho}(1) = \sigma(1)$;

(iii) the method is stable, i.e. all roots of $\rho(\zeta)$ are inside the closed unit disc;

(iv) and the roots on the unit circle are simple.

3.2.1 Applying the Linear Multistep Method

We wish to apply the linear multistep method $(\rho,\ \sigma)$ to the Volterra integral equations and Volterra integro-differential equations of the second kind to get the common discrete form. For the convergence and stability properties of this method see more in [33], [34], [42].

Let us rewrite (3.1), where $t_0 = 0$, as

$$y(t) = G_n(t) + \int_{t_n}^{t} K(t,s)\ y(s)ds, \qquad t \geq t_n,$$

where

$$G_n(t) = g(t) + \int_{0}^{t_n} K(t,s)\ y(s)ds.$$

At the grid-points we obtain

$$y(t_n) = G_n(t_n).$$

Now consider the following extension of the method $(\rho,\ \sigma)$ to Volterra equations.

Let $\hat{G}_{-k}(t), \ldots, \hat{G}_{-1}(t)$ denote given approximations to $G_{-k}(t), \ldots, G_{-1}(t)$ and define $\hat{G}_n(t)$ [as approximation to $G_n(t)$] for $n \geq 0$ by

$$\sum_{j=0}^{k} \alpha_j \hat{G}_{n+j-k} = h \sum_{j=0}^{k} \beta_j K(t, t_{n+j-k}) y_{n+j-k}\ , \qquad t \geq t_n, \tag{3.30}$$

$$y_n = \hat{G}_n(t_n).$$

We shall always assume that the starting functions $\hat{G}_{-k}(t), \ldots, \hat{G}_{-1}(t)$ are defined by a quadrature

$$\hat{G}_n(t) = g(t) + h\sum_{j=-k}^{-1} w_{nj} K(t,t_j)\, y_j\ , \qquad -k \le n \le -1,\ t \ge t_0, \tag{3.31}$$

where $y_{-k}, \ldots, y_{-1}$ are given starting values.

Lemma 3.2.1 *The linear multistep method (3.30) with starting functions (3.31) can be rewritten as a quadrature method*

$$y_n = h_n(t) + h\sum_{j=0}^{n} \omega_{n-j} K(t,t_j)\, y_j, \qquad n \ge 0, \tag{3.32}$$

where

$$h_n(t) = g(t_n) + h\sum_{j=-k}^{-1} w_{nj} K(t,t_j)\, y_j$$

and the weights ω_n *and* w_{nj} $(n \ge 0, \quad -k \le j \le -1\ )$ *are bounded. Moreover, the weights* ω_n *are the coefficients of the power series*

$$\omega(\zeta) := \sum_{j=0}^{\infty} \omega_n \zeta^n = \frac{\breve{\sigma}(\zeta)}{\breve{\rho}(\zeta)},$$

where $\breve{\rho}(\zeta) = \zeta\rho(\zeta^{-1}) = \alpha_0\zeta^k + \cdots + \alpha_{k-1}\zeta + \alpha_k$ *and* $\breve{\sigma}(\zeta) = \zeta\sigma(\zeta^{-1}) = \beta_0\zeta^k + \cdots + \beta_{k-1}\zeta + \beta_k$.

Proof. Let $\mathrm{K}_n = K(\cdot, t_n) y_n$ $(n \ge -k\ )$. In terms of the generating formal power series

$$\hat{G}(\zeta) = \sum_{0}^{\infty} \hat{G}_n \zeta^n, \qquad K(\zeta) = \sum_{0}^{\infty} K_n \zeta^n,$$

the methods (3.30) reads

$$\breve{\rho}(\zeta)\hat{G}(\zeta) = h\breve{\sigma}(\zeta)K(\zeta) + P(\zeta), \tag{3.33}$$

where

$$\begin{aligned} P(\zeta) \;&=\; (h\beta_0 K_{-1} - \alpha_0 \hat{G}_{-1})\zeta^{k-1} + (h\beta_1 K_{-1} + h\beta_0 K_{-2} - \alpha_1 \hat{G}_{-1} - \alpha_0 \hat{G}_{-2})\zeta^{k-2} + \cdots \\ &\quad + (h\beta_{k-1} K_{k-1} + \cdots + h\beta_0 K_{-k} - \alpha_{k-1}\hat{G}_{-1} - \cdots - \alpha_0 \tilde{G}_{-k})\zeta^0. \end{aligned}$$

Using (3.31) and

$$\sum_{j=0}^{\infty} \alpha_j = \rho(1) = 0,$$

we obtain

$$\begin{aligned} P(\zeta) \;&=\; h\sum_{-k}^{-1} P_j(\zeta)K_j - \left[\alpha_0 \zeta^{k-1} + (\alpha_0 + \alpha_1)\zeta^{k-2} + \cdots + (\alpha_0 + \cdots + \alpha_{k-1})\zeta^0\right] g \\ &=\; h\sum_{-k}^{-1} P_j(\zeta)K_j + \frac{\breve{\rho}(\zeta)}{1-\zeta} g, \end{aligned}$$

where $P_j(\zeta)$ are polynomials of degree at most $k-1$ whose coefficients depend only on the method $(\rho,\ \sigma)$ and the starting quadratures w_{nj} $(-k \le n,\ j \le -1\ )$. Hence (3.33) gives

$$\hat{G}(\zeta) = \frac{1}{1-\zeta} g + h\sum_{j=-k}^{-1} \frac{P_j(\zeta)}{\breve{\rho}(\zeta)} K_j + h\frac{\breve{\sigma}(\zeta)}{\breve{\rho}(\zeta)} K(\zeta).$$

We put

$$\frac{P_j(\zeta)}{\breve{\rho}(\zeta)} = \sum_{j=0}^{\infty} w_n \zeta^n, \qquad\qquad \frac{\breve{\sigma}(\zeta)}{\breve{\rho}(\zeta)} = \sum_{j=0}^{\infty} \omega_n \zeta^n, \tag{3.34}$$

and observe that

$$\frac{1}{1-\zeta} = \sum_{0}^{\infty} \zeta^n.$$

So we have

$$\hat{G}_n = g + h\sum_{j=-k}^{-1} w_{nj}K_j + h\sum_{j=0}^{n} \omega_{n-j}K_j.$$

Evaluation of this identity at t_n yields (3.32).

Further, the coefficients of

$$\frac{1}{\breve{\rho}(\zeta)} = \sum_{0}^{\infty} r_n\zeta^n$$

satisfy a linear recurrence relation with characteristic polynomial $\rho(\zeta)$. Hence the stability of the method (2.29) implies the boundedness of ω_n and w_{nj}. ■

The special structure of the weights in (3.32) makes linear multistep methods particularly suitable for the treatment of convolution equations

$$y(x) = f(x) + \int_0^x k(x-s)y(s)ds \ , \qquad x \geq 0 \ .$$

Using (3.32), we have

$$y_n = f_n + h\sum_{j=0}^{n} \omega_{n-j}k_{n-j}yj, \qquad n \geq 0, \tag{3.35}$$

where

$$k_n = k(nh) \quad \text{and} \quad f_n = f(t_n) + h\sum_{j=-k}^{-1} w_{nj}K_{n-j}\, y_j.$$

Now, equation (3.35) is a discrete Volterra equation of convolution type.

3.2.2 Construction of Quadrature Rules

Consider the use of $(\rho, \ \sigma)$-reducible quadrature rules; that is, rules which are constructed by means of linear multistep (LM) methods for ordinary differential equations. It is well known that certain LM methods, such as the Adams-Bashforth-Moulton or the Nystrom-Milne-Simpson formulae are derived from interpolatory quadrature see [17], so that the relationship between such LM methods and quadrature rules is quite natural. For the construction of highly stable methods for Volterra integral equations and Volterra integro-differential equations, one should choose quadrature rules generated by highly stable LM methods for ordinary differential equations.

The application of a LM method with real coefficients α and β, defined in (3.28), to the quadrature problem

$$I'(x) = \phi(x), \qquad I(x_0) = 0 \tag{3.36}$$

yields the relations

$$\sum_{i=0}^{k} \alpha_i I_{n-i} = h \sum_{i=0}^{k} \beta_i \phi(x_{n-i}), \qquad n \geq k, \tag{3.37}$$

for the values of I_n approximating

$$I(x_n) = \int_{x_0}^{x_n} \phi(y) dy.$$

If we assume that the starting values $I_0, \ldots, I_{n-1}$ are defined by

$$I_n = h \sum_{j=0}^{k-1} w_{nj}^{(s)} \phi(x_j), \qquad n = 0, 1, \ldots, k-1, \tag{3.38}$$

where the superscript (s) indicates that we are dealing with starting values, then I_n defined in (3.37) depends linearly on $\phi(x_0), \ldots, \phi(x_n)$. To be specific,

$$I_n = h\sum_{j=0}^{n} w_{nj}\phi(x_j), \qquad n \geq k, \tag{3.39}$$

where the weights w_{nj} satisfy the relations

$$\sum_{i=0}^{k}\alpha_i w_{n-i,j} = \left\{\begin{matrix} 0, & \text{for } j = 0, 1, \ldots, n-k-1 \\ \beta_{n-j}, & \text{for } j = n-k, n-k+1, \ldots, n \end{matrix}\right\} \quad n \geq k, \tag{3.40}$$

here we have defined $w_{nj} = w_{nj}^{(s)}$ for $n, j = 0, 1, \ldots, k-1$ and $w_{nj} = 0$ for $j > \max\{n, k-1\}$. The rules (3.38) employing the weights $\left\{w_{nj}^{(s)}\right\}_{n,j=0}^{k-1}$ are called starting quadrature rules, denoted by F_k. By means of the relations (3.40) the weights w_{nj} can be generated provided that the starting quadrature F_k and the LM coefficients α_i and β_i are given. Arranging the weights w_{nj} in a matrix of the form

$$W = \begin{bmatrix} w_{00} & \cdots & w_{0,k-1} & & & \\ \vdots & \ddots & \vdots & & 0 & \\ w_{k-1,0} & \cdots & w_{k-1,k-1} & & & \\ w_{k0} & \cdots & w_{k,k-1} & w_{kk} & & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ w_{n,0} & \cdots & w_{n,k-1} & w_{n,k} & \cdots & w_{n,n} \\ \vdots & \ldots\ldots & \vdots & \vdots & \ldots\ldots & \vdots \end{bmatrix} = \begin{bmatrix} F_k & 0 \\ W_k & \Omega \end{bmatrix}, \tag{3.41}$$

we observe from (3.40) that the entry w_{nj} in the jth column of W depends only on $w_{n-k,j}, \ldots, w_{n-1,j}$. As a consequence, only the weights in the matrix W_k in (3.41) depend on the entries of F_k . The remaining weights are independent of the starting quadrature rules. Moreover, due the presence of zero entries in the upper-triangular part of W , one can easily derive from (3.40) with $j \geq k$ that Ω is a matrix of the form

$$\Omega = \begin{bmatrix} \omega_0 & 0 & 0 & 0 & 0 \\ \omega_1 & \omega_0 & 0 & 0 & 0 \\ \omega_2 & \omega_1 & \omega_0 & 0 & 0 \\ \vdots & \omega_2 & \omega_1 & \omega_0 & 0 \\ & \cdots & \ddots & \ddots & \ddots \end{bmatrix} \tag{3.42}$$

where the sequence $\{\omega_n\}_{n=0}^{\infty}$ satisfies

$$\begin{array}{lllll} \alpha_0\omega_0 & & & & = \beta_0 \\ \alpha_0\omega_1 & + \ \alpha_1\omega_0 & & & = \beta_1 \\ & \cdots & & & \cdots \\ \alpha_0\omega_k & + \ \alpha_1\omega_{k-1} & \cdots & + \ \alpha_k\omega_0 & = \beta_k \end{array} \tag{3.43}$$

$$\alpha_0\omega_n \ + \alpha_1\omega_{n-1} \ \cdots \ + \alpha_k\omega_{n-k} \ = 0 \ , \qquad n \geq k+1. \tag{3.44}$$

From (3.41) and (3.42) it is obvious that $w_{nj} = \omega_{n-j}$ for $n - j \geq 0, \quad j \geq k$.

Thus, for construction of the quadrature weights W it is sufficient to generate the first k columns by means of (3.40) and to generate the sequence$\{\omega_n\}$ by means of (3.43) and (3.44).

Finally, we introduced discreet form of continuous case using different numerical methods. We have established also programs that solve the continuous case using the weights generated by those methods.

Chapter 4

Estimation of Solutions of Volterra Equations

The estimates of the solutions for both linear and nonlinear Volterra equations will be discussed, using some comparison theorems and auxiliary formula for representations of the solutions. Under some assumption with respect to the kernel $K(t)$, the estimates of the solutions do not depend on the kernel and defined only by the properties of the perturbation of $f(t)$.

4.1 Estimation of Solutions of Linear Volterra Equations

Consider the linear scalar Volterra equation

$$x_n = \sum_{j=1}^{n} K(n,j)x_j + f_n, \qquad n \geq 1. \tag{4.1}$$

For the estimates of the solutions of the linear scalar Volterra equation we have the following theorem.

Theorem 4.1.1 *Let the function* $K(n,j)$ *be nonpositive and nondecreasing with* *respect to n, $n \geq j$ and the perturbations f_n satisfy the condition*

$$\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty. \tag{4.2}$$

Then the solution of (4.1) satisfies the estimates ($f_0 = 0$)

$$\sum_{j=1}^{n} (f_j - f_{j-1})_- \leq x_n \leq \sum_{j=1}^{n} (f_j - f_{j-1})_+. \tag{4.3}$$

Here, $f_+ = \max(0, f) = \frac{1}{2}(|f| + f)$ and $f_- = \min(0, f) = f - f_+$.

Proof. Let us represent (4.1) in the form

$$x_n - \sum_{j=1}^{n} K(j,j) x_j = \sum_{j=1}^{n} \left[K(n,j) - K(j,j) \right] x_j + f_n. \tag{4.4}$$

Using mathematical induction, solutions x_n of (4.4) can be represented as

$$\begin{aligned} x_n \quad = \quad & \sum_{j=1}^{n} \frac{f_j - f_{j-1}}{(1 - K(n,n)) \cdots (1 - K(j,j))} \\ & + \sum_{j=1}^{n} \frac{1}{(1 - K(n,n)) \cdots (1 - K(j,j))} \\ & \times \sum_{l=1}^{j-1} \left[K(j,l) - K(j,j) \right] x_l. \end{aligned} \tag{4.5}$$

Here, $f_0 = 0$ and double sum at the right-hand side is equal zero for $j = 1$.

It is clear that, for $n = 1$ the value x_1 is a solution of (4.4). Assume now that formula (4.5) is valid for some $n > 1$ and prove its validity for $n + 1$. For this purpose using (4.1) we obtain

$$\begin{aligned} x_{n+1}-x_n &= K(n+1,n+1)x_{n+1}+f_{n+1}-f_n \\ &\quad +\sum_{j=1}^{n}\left[K(n+1,j)-K(n,j)\right]x_j. \end{aligned} \tag{4.6}$$

Let us divide both sides of (4.6) by $1-K(n+1,n+1)$. Then, taking into account the validity of the representation (4.5) for x_n, we must conclude that x_{n+1} can also be written in the form (4.5) for x_n and relations $K(n+1,j)-K(n,j)\geq 0$, it follows that

$$\begin{aligned} (x_n)_+ &\leq \sum_{j=1}^{n}\frac{(f_j-f_{j-1})_+}{(1-K(n,n))\cdots(1-K(j,j))} \\ &\quad +\sum_{j=1}^{n}\frac{1}{(1-K(n,n))\cdots(1-K(j,j))} \\ &\quad \times\sum_{l=1}^{j-1}\left[K(j,l)-K(j-1,l)\right](x_l)_+ . \end{aligned} \tag{4.7}$$

Here, $f_0=0$ and double sum is equal zero for $j=1$.

Let us take any $u_1\geq (x_1)_+$ and denote by u_n the solution of the equation

$$\begin{aligned} u_n &= \sum_{j=1}^{n}\frac{(f_j-f_{j-1})_+}{(1-K(n,n))\cdots(1-K(j,j))} \\ &\quad +\sum_{j=1}^{n}\frac{1}{(1-K(n,n))\cdots(1-K(j,j))} \\ &\quad \times\sum_{l=1}^{j-1}\left[K(j,l)-K(j-1,l)\right]u_l. \end{aligned} \tag{4.8}$$

This equation is of the general form

$$u_n = a_n + \sum_{j=1}^{n} b_j u_j.$$

Here, the b_j , $(j = 1, 2, ..., n)$ are nonnegative since all coefficients $(1 - K(j, j))$, $[K(j, l) - K(j - 1, l)]$ of u_l in the right-hand side of (4.8) are nonnegative . Then, we can use a comparison theorem see [31] and conclude that

$$(x_n)_+ \leq u_n.$$

Now, let us transform the equation for u_n as follows. Note that u_{n+1}, by virtue of (4.8), can also be represented in the form

$$\begin{aligned} (1 - K(n+1, n+1))u_{n+1} &= (f_{n+1} - f_n)_+ \\ &+ \sum_{j=1}^{n} \frac{(f_j - f_{j-1})_+}{(1 - K(n, n)) \cdots (1 - K(j, j))} \\ &+ \sum_{j=1}^{n} [K(n+1, j) - K(n, j)]\, u_j \\ &+ \sum_{j=1}^{n} \frac{1}{(1 - K(n, n)) \cdots (1 - K(j, j))} \\ &\times \sum_{l=1}^{j-1} [K(j, l) - K(j - 1, l)]\, u_l. \end{aligned}$$

Form this and the relation (4.8) for u_n, it follows that

$$u_{n+1} = K(n+1, n+1)u_{n+1} + (f_{n+1} - f_n)_+ + u_n + \sum_{j=1}^{n} [K(n+1, j) - K(n, j)]\, u_j. \quad (4.9)$$

Let us use once more the mathematical induction method. For $n = 1$ and $n = 2$

we have by virtue of (4.8)

$$u_1 = K(1,1)u_1 + (f_1)_+ \,,$$

$$u_2 = K(2,2)u_2 + K(1,1)u_1 + (f_2 - f_1)_+ + (f_1)_+ \,.$$

Now let us assume that the relation for u_n is valid for some $n \geq 2$ and show that it will also be valid for $n+1$. According to our assumption, we have

$$u_n = \sum_{j=1}^{n} K(n,j)u_j + \sum_{j=1}^{n} (f_j - f_{j-1})_+ \,. \tag{4.10}$$

If we substitute, at the right-hand side of (4.9), expression (4.10) instead of u_n, then we obtain

$$u_{n+1} = \sum_{j=1}^{n+1} K(n+1,j)u_j + \sum_{j=1}^{n+1} (f_j - f_{j-1})_+ \,.$$

From this, it follows that the function u_n satisfies (4.10) for all $n \geq 1$. But, according to our conditions, the kernel $K(n,j)$ defined for $n \geq j \geq 1$, is nonpositive and also $u_n \geq (x_n)_+ \geq 0$. From this and representation (4.10), we conclude that

$$u_n \leq \sum_{j=1}^{n} (f_j - f_{j-1})_+ \,.$$

Therefore,

$$x_n \leq (x_n)_+ \leq \sum_{j=1}^{n} (f_j - f_{j-1})_+ \,.$$

In order to obtain the estimate of the solution x_n of (4.1) from below it is sufficient to introduce new variable $y_n = -x_n$. As a result we obtain

$$\sum_{j=1}^{n}(f_j - f_{j-1})_- \le (x_n)_- \le x_n.$$

This completes the proof. ■

4.2 Estimation of Solutions of Nonlinear Volterra Equations

Consider Volterra scalar nonlinear difference equation

$$x_n = \sum_{j=1}^{n} F(n, j, x_j) + f_n, \qquad n \ge 1. \tag{4.11}$$

Theorem 4.2.1 *Let us assume that, in (4.11), f_n is a given sequence satisfying condition*

$$\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty, \tag{4.12}$$

and $F(n, j, x)$ is a function with the following properties:

i) It continuous with respect to the third argument;

ii) It nonpositive for all $x \in (-\infty, \infty)$;

iii) It nonincreasing with respect to n, $n \ge j$, for all $x \in (-\infty, \infty)$. Then, the solutions x_n of (4.11) satisfy the inequalities (which do not depend on F)

$$f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \le 2x_n \le f_n + |f_1| + \sum_{j=1}^{n-1} |f_{j+1} - f_j|, \qquad n \ge 1. \tag{4.13}$$

It is assumed that in (4.13) both sums are equal to zero for $n = 1$.

Proof. Let us introduce the function u_n defined by

$$\begin{aligned} u_n &= \sum_{j=1}^{n} F_+(n,j,x_j) + \frac{1}{2}\left(f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j|\right) - \epsilon n \\ &= \sum_{j=1}^{n} F_+(j,j,x_j) + \sum_{j=1}^{n} \left(F_+(n,j,x_j) - F_+(j,j,x_j)\right) \\ &\quad + \frac{1}{2}\left(f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j|\right) - \epsilon n. \end{aligned}$$

Here, $F_+ = \max(0, F)$ and $\epsilon > 0$ is some number. First of all, let us show that the function

$$F_+(n,j,x_j) - F_+(j,j,x_j)$$

is nonincreasing with respect to n, $n \geq j$, and nonpositive. Because of the inequality $xF(n,j,x) \leq 0$ for all $x \in (-\infty, \infty)$, we have

$$F(n,j,x) \geq 0 \quad \text{for } x \leq 0 \quad \text{and} \quad F(n,j,x) \leq 0 \quad \text{for } x \geq 0. \tag{4.14}$$

Further, by virtue of the conditions of the Theorem,

$$x\,[F(n+1,j,x) - F(n,j,x)] \geq 0, \qquad x \in (-\infty, \infty).$$

Therefore,

$$\begin{aligned} F(n+1,j,x) - F(n,j,x) &\leq 0 \qquad \text{for } x \leq 0, \\ F(n+1,j,x) - F(n,j,x) &\leq 0 \qquad \text{for } x \geq 0. \end{aligned}$$

From this and (4.14), it follows that

$$\begin{aligned} F_+(l+1,j,x) - F_+(l,j,x) &\leq 0, \qquad x \leq 0, \\ F_+(l+1,j,x) - F_+(l,j,x) &= 0, \qquad x \geq 0,\ l \geq j. \end{aligned}$$

Let us sum both parts of the last relations with respect to l from$l = j$ to $l = n-1$. As a result, we obtain

$$F_+(n,j,x) - F_+(j,j,x) \leq 0, \qquad x \leq 0,$$

$$F_+(n,j,x) - F_+(j,j,x) = 0, \qquad x \geq 0,\ n \geq j. \tag{4.15}$$

Hence, the sum

$$\sum_{j=1}^{n} (F_+(n,j,x_j) - F_+(j,j,x_j)) \tag{4.16}$$

is nonincreasing with respect to n and also nonpositive by virtue of (4.15).

Let us show further that $u_1 < 0,$ by considering the different signs of f_1. From the definition of the function $u_n,$ we get

$$u_1 = F_+(1,1,x_1) + \frac{1}{2}(f_1 - |f_1|) - \epsilon.$$

Case 1: $f_1 > 0$. From (4.14), any root of the equation

$$x_1 = F(1,1,x_1) + f_1$$

must satisfy the condition $0 \le x_1 \le f_1$. Hence,

$$F_+(1,1,x_1) = 0.$$

Consequently,

$$u_1 = -\epsilon < 0 \qquad \text{for} \;\; f_1 > 0.$$

Case 2: $f_1 < 0$. Any root of the equation

$$x_1 = F(1,1,x_1) + f_1$$

must satisfy the condition $f_1 \le x_1 \le 0$. Hence,

$$F(1,1,x_1) = F_+(1,1,x_1)$$

and also

$$-f_1 \ge F(1,1,x_1) \ge 0.$$

So, in this case

$$u_1 \le -f_1 + \frac{1}{2}\left(f_1 - |f_1|\right) - \epsilon = -\epsilon < 0, \qquad \text{for } f_1 < 0.$$

Case 3: $f_1 = 0$. If $f_1 = f_2 = \cdots = f_m = 0$, then it would be sufficient to come to the first nonzero number f_j.

Let us denote $F_- = \min(0, F)$ and introduce one more function υ_n, defined by the relation

$$\begin{aligned}
\upsilon_n &= -\sum_{j=1}^{n} F_-(j,j,x_j) - \sum_{j=1}^{n} \left(F_-(n,j,x_j) - F_-(j,j,x_j)\right) \\
&\quad -\frac{1}{2}\left(f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j|\right) - \epsilon n,
\end{aligned}$$

for $n \geq 1,\ \epsilon > 0$. Let us check that $\upsilon_1 < 0,$ by using the same arguments as above for the proof that $u_1 < 0$. We have

$$\upsilon_1 = -F_-(1,1,x_1) - \frac{1}{2}\left(f_1 - |f_1|\right) - \epsilon.$$

If $f_1 > 0,$ then any root of the equation

$$x_1 = F(1,1,x_1) + f_1$$

must satisfy the condition $0 \leq x_1 \leq f_1$. Hence,

$$F_-(1,1,x_1) = F(1,1,x_1)$$

and, moreover,

$$-f_1 \leq F_-(1,1,x_1) \leq 0, \qquad \text{i.e.,}\ \ 0 \leq -F_-(1,1,x_1) \leq f_1.$$

Therefore,

$$\upsilon_1 \leq -f_1 + \frac{1}{2}\left(f_1 - |f_1|\right) - \epsilon = -\epsilon < 0.$$

If $f_1 < 0$, then any root of the equation

$$x_1 = F(1,1,x_1) + f_1$$

will satisfy the condition $f_1 \leq x_1 \leq 0$, i.e., $F_-(1,1,x_1) = 0$. It means that $\upsilon = -\epsilon < 0$. As a result, for all cases, $\upsilon_1 = -\epsilon < 0$. Let us show that the function

$$F_-(n,j,x_j) - F(j,j,x_j),$$

is a nondecreasing function of n. Since the function $xF(n,j,x)$ is nondecreasing for all $x \in (-\infty, \infty)$, we can conclude that

$$x\left[F(n+1,j,x) - F(n,j,x)\right] \geq 0.$$

Therefore,

$$F(n+1,j,x) - F(n,j,x) \geq 0, \qquad x \geq 0.$$

Besides,

$$F_-(n,j,x) = F(n,j,x), \qquad x \geq 0.$$

It means that

$$F_-(n+1,j,x) - F_-(n,j,x) \geq 0.$$

From this, it follows that the function

$$F_-(n,j,x) - F_-(j,j,x) \geq 0, \quad n \geq j,$$

is a nonnegative and nondecreasing function of n, $n \geq j$. Hence, the sum

$$\sum_{j=1}^{n} [F_{-}(n,j,x) - F_{-}(j,j,x)] \tag{4.17}$$

is nondecreasing with respect to n and nonnegative. Let us remark also that, form the definitions of the functions u_n, υ_n and (4.11), it follows that

$$x_n = \sum_{j=1}^{n} (F_{+}(n,j,x_j) - F_{-}(n,j,x_j)) + f_n = u_n - \upsilon_n. \tag{4.18}$$

Further, the sum (4.16) is nonincreasing with respect to n , then because of (4.18), we get

$$\begin{aligned} u_{n+1} - u_n &\leq F_{+}(n+1, n+1, x_{n+1}) - \epsilon + \frac{1}{2}\left(f_{n+1} - f_n - |f_{n+1} - f_n|\right) \\ &\leq F_{+}(n+1, n+1, x_{n+1}) - \epsilon \\ &= F_{+}(n+1, n+1, u_{n+1} - \upsilon_{n+1}) - \epsilon. \end{aligned} \tag{4.19}$$

Similarly, because the sum (4.17) is nondecreasing with respect to n , we obtain for the difference $\upsilon_{n+1} - \upsilon_n$ the estimate

$$\begin{aligned} \upsilon_{n+1} - \upsilon_n &\leq -F_{-}(n+1, n+1, x_{n+1}) - \epsilon \\ &= -F_{-}(n+1, n+1, u_{n+1} - \upsilon_{n+1}) - \epsilon. \end{aligned} \tag{4.20}$$

As a result, we have obtained that $u_1 < 0$, $\upsilon_1 < 0$ and both inequalities (4.19), (4.20) are valid. Now, we have three possibilities.

Case 1: If $u_1 < 0$, $\upsilon_1 < 0$ for all $n \geq 1$ then by virtue of (4.18) and conditions

$$F_+(n, j, x_j) \geq 0, \qquad F_-(n, j, x_j) \leq 0,$$

we obtain

$$\begin{aligned}\frac{1}{2}\left(f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j|\right) - \epsilon n &\leq u_n \leq u_n - \upsilon_n = x_n \leq \upsilon_n \\ &\leq \frac{1}{2}\left(f_n + |f_1| + \sum_{j=1}^{n-1} |f_{j+1} - f_j|\right) + \epsilon n.\end{aligned}$$

From this and arbitrariness of $\epsilon > 0$, we derive the estimates (4.13).

Now consider two cases when inequalities $u_n < 0$ and $\upsilon_n < 0$ are not valid for all $n > 1$.

Case 2: Assume that there exists a first moment $n_0 > 1$ such that

$$u_n < 0, \qquad 1 \leq n \leq n_0, \qquad u_{n_0+1} \geq 0, \qquad \upsilon_n \leq 0, \qquad 1 \leq n \leq n_0,$$

but $u_{n_0+1} - \upsilon_{n_0+1} \geq 0$. Then we have

$$u_{n_0+1} - u_{n_0} \geq 0.$$

On the other hand, by virtue of (4.19) and (4.14)

$$u_{n_0+1} - u_{n_0} \leq F_+(n_0 + 1, n_0 + 1, u_{n_0+1} - \upsilon_{n_0+1}) - \epsilon = -\epsilon < 0.$$

Case 3: Consider now the second possible case when there exists a first moment

$n_0 > 1$ such that

$$\upsilon_n < 0, \qquad 1 \leq n \leq n_0, \qquad \upsilon_{n_0+1} \geq 0, \qquad u_n \leq 0, \qquad 1 \leq n \leq n_0,$$

and besides $\upsilon_{n_0+1} - u_{n_0+1} \geq 0$. Then we have

$$\upsilon_{n_0+1} - \upsilon_{n_0} \geq 0.$$

On the other hand, due to (4.20) and (4.14)

$$\upsilon_{n_0+1} - \upsilon_{n_0} \leq -F_-(n_0+1, n_0+1, u_{n_0+1} - \upsilon_{n_0+1}) - \epsilon = -\epsilon.$$

Since Cases 2 and 3 lead to contradictions, the remaining Case 1 proves the estimates (4.13) and proof of the theorem complete. ■

Chapter 5

Conclusion and Further Research

We introduced linear difference equations, which occur in numerous settings and forms, both in mathematics itself and its applications. Mainly we concentrated on Volterra difference equations of the second type.

Volterra difference equations of the second type have been discussed in many aspects. With different types of kernels, stability, periodicity, and boundedness have been studied. Results on the boundedness and stability of solutions of these equations based on the growth properties of the fundamental matrix and the resolvent matrix. Using these properties we found an explicit criteria. In the coming days, we expect generalization of some results work on the implicit form. Also, nonhomogeneous cases of Volterra type will be discussed in terms of their perturbations.

Also, how to change continuous case to a discrete case presented through out numerical methods. Specific programs have been constructed with different methods that solve both Volterra integrals and intgro-differential equations. For the future, we should look for quadrature rules generated by specific weights to construct highly stable methods.

We defined estimations of the solutions, of both cases linear and nonlinear not depending on the kernel. These estimations are defined only by the properties of the perturbation terms.

In conclusion we suggest the construction of an algorithm that specify all information that are needed, stability, periodicity, boundedness, and estimations, by simply entering kernel and perturbation function of Volterra difference equations of the second kind.

Appendix A

A program for solving VIE

A program for solving VIE using BE, TR, and ST weights in MATLB format as m-files.

```
function s = volterraw (method,t0,y0,tn,n)
%this function volterrawe evaluate the Volterra Intgeral Equation using Quadrature weights
% method =1 using Backward Euler method of order1
%method = 2 using Trapezoidal method of order 2
% n is the number of steps
% t0 is the starting pointt(0)
% y0 the value of the function at t(0)
% tn the end point t(n)
h = (tn-t0)/n;
for e = 1:n
t(e)=t0+e*h;
end
if method == 1
  % this part will generate weights using Backward Euler method of order 1
  disp('Backward Euler method of order 1')
  for l =1:n+1
```

```
          for g =1:n+1
                if   l >=g
                    w(l,g)=1;
                else
                   w(l,g)=0;
                end
          end
      end
   else  if  method ==2
        % this part will generate weights using Trapezoidal method of order 2
       disp('Trapezoidal method of order 2')
       for  l = 1:n+1
          for  g =1:n+1
             if  l == j
                 w(l,g) = 1/2;
              elseif  l >g
               w(l,g) =1;
              else
                 w(l,g)=0;
             end
           end
        end
   else  if  method ==3
           % this part will generate weights using Simpson + Trapezoidal method of
order 3
```

```
disp('Simpson + Trapezoidal method of order 3')
 for r =1:n+1
    for d =1:n+1
         if d >r
             w(r,d)=0;
         else
             w(1,1)=1/3;
             w(2*r,2*r) =1/2;
              w(2*r+1,2*r+1) =1/3;
             w(2,1)=1/3+1/2;
              w(2+2*r,1+2*r)=1/3+1/2;
             w(3,2)=4/3;
             w(3+2*r,2+2*r)=4/3;
          end
    end
  end
 for i =3:n+1
 w(i,1)=1/3+1/3;
     for j=1:i-2
               if rem(j,2) ==0
                      w(i,j)=4/3;
             else
                       w(i,j) =1/3+1/3;
              end
      end
```

```
end
else
% this part will write choose another method i.e. 1 ,2 or 3
disp('choose another method i.e. 1 ,2 or 3')
end
% we will call two external functions:
% k(x,y) and g(x) these two functions will be supplied by the user
y0;
y(1)=(feval('g',t(1))+h*w(2,1)*(feval('k',t0,t0)*y0))/(1-h*w(2,2)*feval('k',t(1),t(1)));
for i =2:n
        s=(feval('g',t(i))+h*w(i+1,1)*(feval('k',t(i),t0)*y0))/(1-h*w(i+1,i+1)
feval('k',t(i),t(i)));
    for j=1:i-1
            y(i)=(h*w(i+1,j+1)*feval('k',t(i),t(j))*y(j))*(1/(1-h*w(i+1,i+1)
feval('k',t(i),t(i))));
    s=s+y(i);
    end
y(i)=s;
end
% this part to compare Exact fun(x) and our Approximation y(i)
error(i)=fun(tn)-y(i);
Exact=fun(tn)
Error=error(i).
```

Appendix B

A program for solving VIDE 1

A program for solving VIDE using (3.23) and Trapezoidal weights in MATLB format as m-files.

```
functions=intgrodiffern1(t0,y0,tn,n)
h=(tn-t0)/n;
for e =1:n
t(e)=t0+e*h;
end
y0;
y(1)=(y0+(h/2)*(feval('gs',t0)+feval('gs',t(1)))+((h^2)/2)*((feval('ks',t0,t0)
        +feval('ks',t(1),t0))*y0))/(1-(((h^2)/2))*feval('ki',t(1),t(1)));
for i =2:n
    s=(y(i-1)+(h/2)*(feval('gs',t(i-1))+feval('gs',t(i)))+(((h^2)/2)*(feval('ks',t(i-
1),t0)
        +feval('ks',t(i),t0))*y0))/(1-(((h^2)/2)*feval('ks',t(i),t(i))));
    for j =1:i-1
        y(i)=(((h^2)/2)*(feval('ks',t(i-1),t(j))+feval('ks',t(i),t(j)))*y(j))
            /(1-(((h^2)/2)*feval('ks',t(i),t(i))));
    s=s+y(i);
    end
```

```
y(i)=s;
end
EXACT=funs(tn)
error=EXACT-s.
```

Appendix C

A program for solving VIDE 2

A program for solving VIDE using (3.25) and Simpson's weights in MATLAB formate as m-files.

```
functions =intgrodiffern2(t0,y0,y1,tn,n)
%this function intgro evaluate the Volterra Intgero-differential Equation using BE weights
% t0 is the starting point t(0)
% y0 the value of the function at t(0)
%y1 the value of the function at t(1)
% tn the end point t(n)
% n is the number of steps
h=(tn-t0)/n;
for e=1:n+5;
    t(e)=t0+e*h;
end
y0;
y(1)=y1;
y(2)=(1/(1-((h^2)/3)*feval('ks',t(2),t(2))))*(y0+((h/3)*(feval('gs',t0)
 +4*feval('gs',t(1))+feval('gs',t(2))))+((h/3)*h*(4*feval('ks',t(1),t(1))
 +feval('ks',t(2),t(1)))*y(1))+((h/3)*h*(feval('ks',t0,t0)+4*feval('ks',t(1),t0)
```

```
 +feval('ks',t(2),t0))*y0));
for i=3:n
        s=(1/(1-((h^2)/3)*feval('ks',t(i),t(i))))*(y(i-2)+((h/3)*(feval('gs',t(i-2))
        +4*feval('gs',t(i-1))+feval('gs',t(i)))+((h/3)*h*(4*feval('ks',t(i-1),t(i-1))
       +feval('ks',t(i),t(i-1)))*y(i-1))+((h/3)*h*(feval('ks',t(i-2),t0)+4*feval('ks',t(i-
1),t0)
         +feval('ks',t(i),t0))*y0)));
    for j=1:i-2
             y(i)=(1/(1-((h^2)/3)*feval('ks',t(i),t(i))))*((h/3)*h*(feval('ks',t(i-
2),t(j))+4*feval('ks',t(i-1),t(j))
               +feval('ks',t(i),t(j)))))*y(j);
        s=s+y(i);
     end
y(i)=s;
end
Exact=funs(tn)
error=funs(tn)-y(i).
```

References

[1] P. Agarwal and Patricia J. Wong, *Advanced Topics in Difference Equations*, Kulwer Academic Puiblishers, 1997.

[2] L. Babai and T. Lengyel, *A convergence for recurrent sequences with applications to the partition lattice*, Analysis **12** (1992), 109–119.

[3] H. Brunner, *Iterated collocation methods and their discretizations for Volterra integral equations*, SIAM J. Numer. Anal. **21** (1984), 1132–1145.

[4] Burden and J. Faires, *Numerical Analysis*, 6. ed., International Thomson Publishing company, Pacific Grove, 1997.

[5] T. A. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[6] C. Corduneanu and I. W. Sandberg, *Volterra Equations and Applications*, 2000.

[7] M. R. Crisci, *Stability analysis of discrete recurrence equations of Volterra type with degenerate kernels*, Journal of Math. Analysis and Applications **162** (1991), 49–62.

[8] M. R. Crisci, V. B. Kolmanovskii, E. Russo, and A.Vecchio, *Boundedness of discrete Volterra equations*, J. Math. Anal. Appl. **211** (1997), 106–130.

[9] ———, *Stability of the difference Volterra equations: Direct Liapunov method and numerical procedure*, Computers Math. Applic. **36** (1998), 77–97.

[10] ———, *Asymptotic properties of solutions of Volterra difference equations with finite linear part*, Dynamic Systems and Application **8** (1999), 1–20.

[11] L. M. Delves and J. L. Mohamed, *Computational Methods for Integral Equations*, Combridge University Press, 1985.

[12] S. Elaydi, *Periodicity and stability of linear Volterra difference systems*, J. Math. anal. Appl. **181** (1994), 483–492.

[13] ______, *An Introduction to Difference Equations*, 2. ed., Springer-Verlag, New York, 1999.

[14] P. W. Eloe, M. N. Islam, and Y. N. Raffoul, *Advences of uniform asymptotic stability in nonlinear Volterra discrete systems*, Journal of Difference Equations, (to be appear).

[15] C. Gear, *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice-Hall, London, 1971.

[16] M. I. Gil and Rigoberto Medina, *Boundedness of solution of matrix nonlinear Volterra difference equations*, **7** (2002), 19–22.

[17] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley & Sons, New York, 1962.

[18] ______, *Error Propagation for Difference Methods*, Robert E. Krieger Publishing company, New York, 1977.

[19] E. Isaacson and H. Keller, *Analysis of Numerical Methods*, John Wiley & Sons, New York, 1966.

[20] M. N. Islam and Y. N. Raffoul, *Stability properties of linear Volterra integro-differential equations with nonlinear perturbation*, (to be appear).

[21] ______, *Uniform asymptotic stability in linear Volterra difference equations*, Panamerican mathematical Journal **11** (2001), 61–73.

[22] A. J. Jerri, *Linear Difference Equations with Discrete Transform Methods*, Kulwer Academic Puiblishers, 1996.

[23] L. Johnson and R. Riess, *Numerical Analysis*, 2. ed., Addison-Wesley Publishing company, New York, 1982.

[24] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, New York, 2001.

[25] T. M. Khandaker and Y. N. Raffoul, *Stability properties of linear Volterra discrete systems with nonlinear perturbation, journal of difference equations and applications*, **8** (2002), 857–874.

[26] V. B. Kolmanovskii, *Asymptotic properties of solutions for discrete Volterra equations*, personal communication.

[27] ______, *On the exponential stability of some Volterra difference equations*, Automation and Remote Control **58** (1997), No. 7.

[28] V. B. Kolmanovskii and N. P. Kosreva, *One method of Liapunov functionals construction*, proceedings of dynamic systems and Applications **3** (2001), 343–350.

[29] V. B. Kolmanovskii and A. I. Mataso, *Duality theory in filtering problem for discrete Volterra equations*, 15th Triennial World Congress, Barcelona, Spain.

[30] V. B. Kolmanovskii and J. P. Richard, *Estimation of solutions of Volterra difference equations*, J. Math. Anal. Appl. **273** (2002), 618–626.

[31] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press, New York, 1988.

[32] H. Lauwerier, *Mathematical Models of Epidemcs*, Math. Centrum, Amsterdam, 1981.

[33] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, 1985.

[34] Ch. Lubich, *On the stability of linear multistep methods for Volterra convolution equations*, IMA Journal of Numerical Analysis **3** (1983), 439–465.

[35] R. Medina, *Boundedness of perturbed Volterra discrete equations*, Proceeding of Dynamic Systems and Application **3** (2001), 417–424.

[36] E. J. Muth, *Transform Methods with Application to Engineering and Operations Research*, Prentice-Hall Inc, London, 1977.

[37] Y. N. Raffoul, *Uniform asymptotic stability in linear Volterra systems with nonlinear perturbation*, Intl. Journal of Difference Equations, (to be appear).

[38] Y. N. Raffoul and Youssef M. Dib, *Boundedness and stability in nonlinear discrete systems with nonlinear perturbation*, Intl. Journal of Difference Equations, (to be appear).

[39] A. Vecchio, *Stability of direct quadrature methods for system of Volterra integral equations*, Dynam. Systems Appl. **7** (1998), 501–518.

[40] ______, *On the resolvent kernel of Volterra discrete equations*, Functional Differential Eqautions **6** (1999), 187–197.

[41] ______, *Volterra discrete equations: summability of the fundamental matrix*, Journal of Numer. Math. **89** (2001), 783–794.

[42] P. H. M. Wolkenfelt, *The construction of reducible quadrature rules for Volterra integral and integro-differential equations*, IMA Journal of Numerical Analysis **2** (1982), 131–152.

[43] S. Zhang, *The unique existence of periodic solutions of linear Volterra difference equations*, J. Math. anal. Appl. **193** (1995), 419–430.

Vita

- Said Ali Al-Garni.

- Born in Thoqba, in March 1974, in Saudi Arabia.

- Received B. S. of Science degree in Mathematics in May 1997.

- Joined Mathematical Science Department as a Graduate Assistant.

- Received M. Sc. of Science Degree in Mathematics in April 2003.

- Homepage: http://users.kfupm.edu.sa/math/garnis/

- E mail: garnis@kfupm.edu.sa